Semiorthogonal decompositions and stability conditions

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The main characters

1. $\mathcal{C}$ = (pre-)triangulated (dg-)category,
   - $\mathrm{D}^b_{\mathrm{coh}}(X)$ for X a smooth projective variety; or
   - $\mathrm{D}^b(\mathrm{mod}\, A)$ for A a finite dimensional algebra.
2. $\mathrm{Stab}(\mathcal{C})$ = complex manifold of stability conditions on $\mathcal{C}$ (see [2])
3. Semiorthogonal decompositions: full triangulated subcategories $\{\mathcal{C}_i\}_{i=1}^n$ of $\mathcal{C}$ that generate $\mathcal{C}$ and s.t. $\mathrm{Hom}(\mathcal{C}_j, \mathcal{C}_i) = 0$ for $i < j$.

Stability conditions [1, 2]

Slicing $\mathcal{P}$ on $\mathcal{C}$: collection of full additive subcat. $\{\mathcal{P}(\phi) : \phi \in \mathbb{R}\}$ s.t.

1. $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$
2. for $\phi_1 > \phi_2$ and $E_i \in \mathcal{P}(\phi_i)$ for $i = 1, 2$, $\mathrm{Hom}_{\mathcal{C}}(E_1, E_2) = 0$
3. for any $E \in \mathcal{C}$, there are maps $0 = E_0 \to E_1 \to \cdots \to E_n = E$ with $F_i = \mathrm{Cone}(E_{i-1} \to E_i) \in \mathcal{P}(\phi_i)$ for $1 \leq i \leq n$ and $\phi_1 > \cdots > \phi_n$.

$E \in \mathcal{P}(\phi)$: *semistable* of phase ϕ; (3) is called : *Harder-Narasimhan (HN) filtration* of E; F_i called the *HN factors* of E.

Prestability condition on $\mathcal{C}$: $\sigma = (Z, \mathcal{P})$ where $Z \in \mathrm{Hom}(K_0(\mathcal{C}), \mathbb{C})$ s.t. $\forall$ $\phi \in \mathbb{R}$ and $E \in \mathcal{P}(\phi)$, $Z(E) = m(E) \cdot \exp(i\pi\phi)$ with $m(E) \in \mathbb{R}_{>0}$ (*mass*)

A *stability condition* on $\mathcal{C}$ also has the support property of [1, 4]. The collection of all stability conditions, $\mathrm{Stab}(\mathcal{C})$, has a canonical *complex manifold structure* [2].

Stability conditions on varieties

- X : complex projective variety, $\mathrm{Stab}(X)$ = stability conditions on $\mathrm{D}^b(X)$ s.t. Z factors through $K_0(X) \xrightarrow{\mathrm{ch}} H^*_{\mathrm{alg}}(X; \mathbb{C})$.
- $\mathrm{Stab}(X) \to \mathrm{Hom}(H^*_{\mathrm{alg}}(X;\mathbb{C}), \mathbb{C})$ given by $(Z, \mathcal{P}) \mapsto Z$ is a covering. $\mathrm{Stab}(X)$ is a (noncompact) $\mathbb{C}$-manifold modeled on $H^*_{\mathrm{alg}}(X;\mathbb{C})^\vee$

Notation

- $\sigma_\bullet$ denotes a path $t \mapsto \sigma_t$ from $[0, \infty) \to \mathrm{Stab}(\mathcal{C})$
- given $E \in \mathrm{Ob}(\mathcal{C})$, $\ell_t(E) := m_{\sigma_t}(E) + i\pi\overline{\phi}_{\sigma_t}(E)$
- $\phi_t^+(E)$ = largest phase of a σ_t-HN factor of E, $\phi_t^-(E)$ is analogous

$\overline{\phi}_{\sigma_t}(E)$ is an average phase function generalizing the phase function of σ_t-semistable objects to *all* objects.

Quasi-convergent paths

1. $0 \neq E \in \mathcal{C}$ is *limit semistable* with respect to $\sigma_\bullet$ if :
$$\lim_{t\to\infty} \phi_t^+(E) - \phi_t^-(E) = 0$$
The class of such objects is denoted $\mathcal{P}_{\sigma_\bullet}$
2. $\sigma_\bullet$ is called *quasi-convergent* if
   (a) all nonzero objects of $\mathcal{C}$ have *limit* HN filtrations with subquotient objects in $\mathcal{P}_{\sigma_\bullet}$; and
   (b) for any $E, F \in \mathcal{P}_{\sigma_\bullet}$, $\ell_t(F) - \ell_t(E)$ either converges as $t \to \infty$ or diverges along a well-defined ray $\mathbb{R}_{>0} \cdot e^{i\theta} \subset \mathbb{C}$.

Subcategories associated to qc. paths

Let $E, F \in \mathcal{P}_{\sigma_\bullet}$; we define total preorders:

1. $E \preceq^{\mathrm{i}} F$ if $\lim_{t\to\infty} \phi_t(F) - \phi_t(E) \leq \infty$
2. $E \preceq F$ if $E \preceq^{\mathrm{i}} F$, and if $E \sim^{\mathrm{i}} F$ then $\lim_{t\to\infty} \log \frac{m_t(E)}{m_t(F)} < \infty$

... and full subcategories of $\mathcal{C}$:

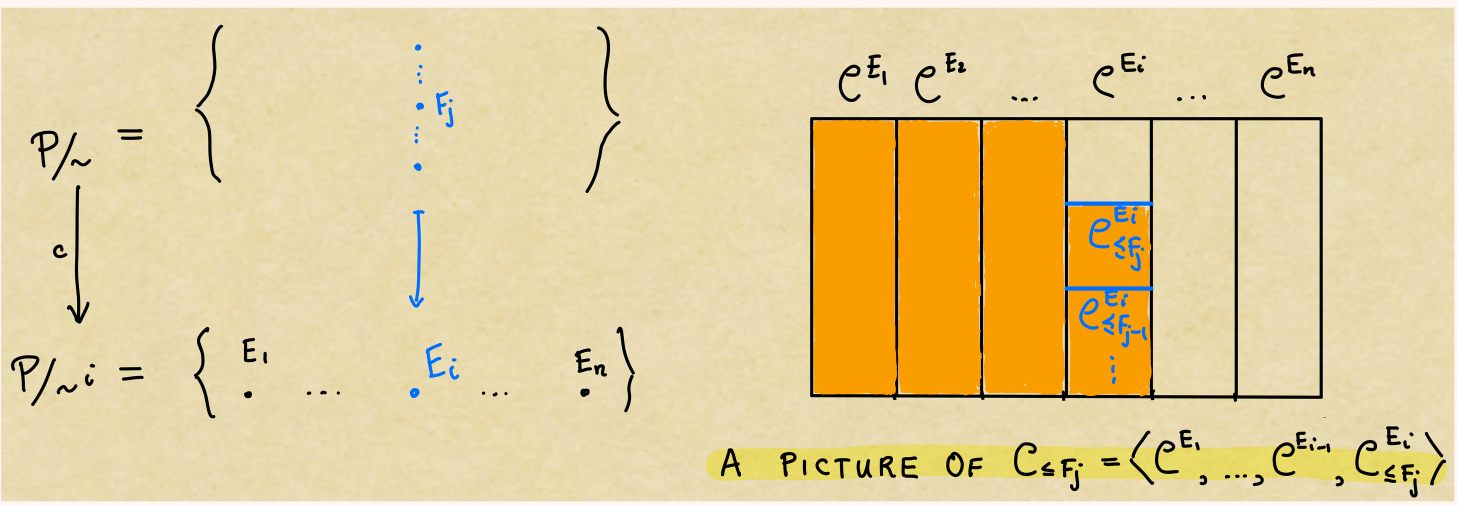
- $\mathcal{C}^E$: with objects with limit HN factors A s.t. $A \sim^{\mathrm{i}} E$.
- $\mathcal{C}_{\preceq F}$: with objects with limit HN factors A s.t. $A \preceq F$
- $\mathcal{C}^E_{\preceq F} = \mathcal{C}^E \cap \mathcal{C}_{\preceq F}$

Some comments:

- $\mathcal{C}^E$ depends only on $[E] \in \mathcal{P}/\sim^{\mathrm{i}}$; $\preceq^{\mathrm{i}}$ is total on $\mathcal{P}/\sim^{\mathrm{i}}$
- $\mathcal{C}_{\preceq F}$ depends only on $[F] \in \mathcal{P}/\sim$; $\preceq$ is total on $\mathcal{P}/\sim$
- $\sim$ refines $\sim^{\mathrm{i}}$, so there is an induced map $c : \mathcal{P}/\sim \to \mathcal{P}/\sim^{\mathrm{i}}$

(A version of) The main results of [5]

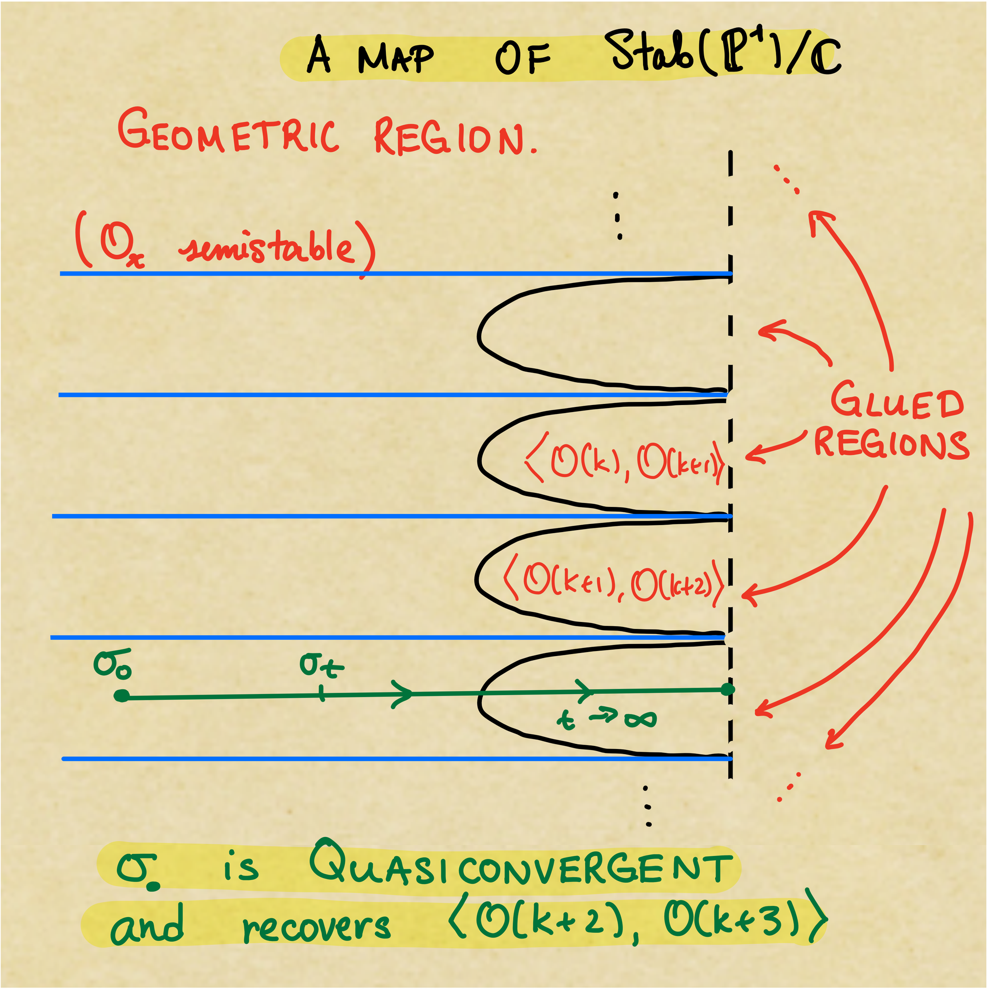
1. $E_1 \prec^{\mathrm{i}} \cdots \prec^{\mathrm{i}} E_n$ a complete set of representatives of $\mathcal{P}/\sim^{\mathrm{i}}$:
$$\exists \text{ semiorthogonal decomposition } \mathcal{C} = \langle \mathcal{C}^{E_1}, \ldots, \mathcal{C}^{E_n} \rangle$$
2. $F_1 \prec \ldots \prec F_k$ a complete set of representatives of $c^{-1}(E_i)$: $\exists$ filt. $\{\mathcal{C}^{E_i}_{\preceq F_1} \subset \mathcal{C}^{E_i}_{\preceq F_2} \subset \cdots \subset \mathcal{C}^{E_i}_{\preceq F_k} = \mathcal{C}^{E_i}\}$ s.t. $\forall\, j$:
$$\mathcal{C}_{\preceq F_j} = \langle \mathcal{C}^{E_1}, \ldots, \mathcal{C}^{E_{i-1}}, \mathcal{C}^{E_i}_{\preceq F_j} \rangle.$$



The main results cont.

3. $\mathcal{C}^{E_i}_{\preceq F_j} / \mathcal{C}^{E_i}_{\preceq F_{j-1}}$ admits a stability condition $\sigma^i_j = (Z^i_j, \mathcal{P}^i_j)$ with
   (a) $\mathcal{P}^i_j = \{\overline{G} : G \in \mathcal{P}_{\sigma_\bullet} \text{ and } G \sim F_j\}$
   (b) $Z^i_j(G) = \lim_{t\to\infty} Z_t(G)/Z_t(F_j)$.
   (c) $\overline{\sigma}^i_j \in \mathrm{Stab}(\mathcal{C}^{E_i}_{\preceq F_j}/\mathcal{C}^{E_i}_{\preceq F_j})/\mathbb{C}$ is independent of choices.
4. Given a nice $\mathcal{C}$ and $\mathcal{C} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_n \rangle$, with $(\tau_i)_{i=1}^n \prod_{i=1}^n \mathrm{Stab}(\mathcal{C}_i)/\mathbb{C}$, $\exists$ q.c. $\sigma_\bullet$ in $\mathrm{Stab}(\mathcal{C})$ recovering $(\langle \mathcal{C}_1, \ldots, \mathcal{C}_n \rangle, (\tau_i)_{i=1}^n)$ using 1 - 3.

The case of $\mathrm{Coh}(\mathbb{P}^1)$ [5, 6]



Further directions

1. (Forthcoming w. D. Halpern-Leistner) construction of a partial compactification of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ in which q.c. $\Rightarrow$ convergent
2. Investigate more examples (beyond projective curves) and paths coming from quantum differential equations on $H^*_{\mathrm{alg}}(X;\mathbb{C})$.

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