Augmented stability conditions

Antonios-Alexandros Robotis

Based on joint works with Daniel Halpern-Leistner and Jeffrey Jiang

Quick review

1. $\mathcal{D} = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for X a complex projective manifold
2. $\mathrm{Stab}(\mathcal{D}) = \mathrm{Stab}(X)$ – space of *stability conditions* $(Z, \mathcal{P})$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$
   - *central charge*: $Z \in \mathrm{Hom}(K_0(X), \mathbf{C})$ which factors through $\mathrm{ch} : K_0(X) \to H^*_{\mathrm{alg}}(X)$.
   - $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbf{R}}$ is a *slicing*, a categorical structure which refines the notion of bounded t-structure
   - $\mathcal{P}(\phi)$ category of *semistable objects* of phase $\phi \in \mathbf{R}$, and

$$Z(E) \in \mathbf{R}_{>0} \cdot \exp(i\pi\phi)$$

   - (*Bridgeland*) $\mathrm{Stab}(X) \to \mathrm{Hom}(H^*_{\mathrm{alg}}(X), \mathbf{C})$ given by $(Z, \mathcal{P}) \mapsto Z$ is a local homeo. $\mathrm{Stab}(X)$ is a $\mathbf{C}$-manifold modeled on $H^*_{\mathrm{alg}}(X; \mathbf{C})^\vee$.
   - Natural $\mathbf{C}$-action on $\mathrm{Stab}(X)$: $w \cdot (Z, \mathcal{P}) = (e^w \cdot Z, \mathcal{P}^w)$.

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Motivation from NMMP

In arXiv:2301.13168, Halpern-Leistner proposes *noncommutative minimal model program (NMMP)*

Heuristic (Optimistic)

Given $\sigma_0 = (Z_0, \mathcal{P}_0) \in \mathrm{Stab}(X)$, solving "canonical ODEs" in $H^*_{\mathrm{alg}}(X;\mathbf{C})^\vee$ with initial point Z_0 gives paths $Z_t : [0,\infty) \to H^*_{\mathrm{alg}}(X;\mathbf{C})^\vee$ which lift to $\sigma_t : [0,\infty) \to \mathrm{Stab}(X)$.

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Quasi-convergent paths

In arXiv:2401.00600, (with D. Halpern-Leistner and J. Jiang) we introduce *quasi-convergent paths* $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})$.

Theorem (HL, J, R '23)

A generic quasi-convergent path σ_t gives a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ plus $\sigma_i \in \mathrm{Stab}(\mathcal{D}_i)/\mathbf{C}$ for $i = 1, \ldots, n$.

1. study growth of $\phi_t(E)$ – if for all $t \gg 0$, $\phi_t(E) < \phi_t(F)$, then $\mathrm{Hom}(F, E) = 0$.
2. $\mathcal{D}_1$ is generated by objects with ϕ_t growing "slowest" and $\mathcal{D}_n$ is generated by objects with ϕ_t growing "fastest."
3. resulting SOD + stability conditions depends only on $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

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Quasi-convergent paths cont.

$\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle + \sigma_i \in \mathrm{Stab}(\mathcal{D}_i)/\mathbf{C}$ for $i = 1, \ldots, n$ is called a *polarised semiorthogonal decomposition*.

Theorem (HL, J, R '23)

Let $\mathcal{D}$ be smooth and proper (as a dg-category). Every polarised SOD $\langle \mathcal{D}_1, \ldots, \mathcal{D}_n | \sigma_1, \ldots, \sigma_n \rangle$ comes from a qc path.

The proof uses the gluing construction of Collins - Polishchuk.

Heuristic

Qc. paths should converge in a (partial) compactification of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ to boundary points which correspond to polarised SODs (+ some additional data!)

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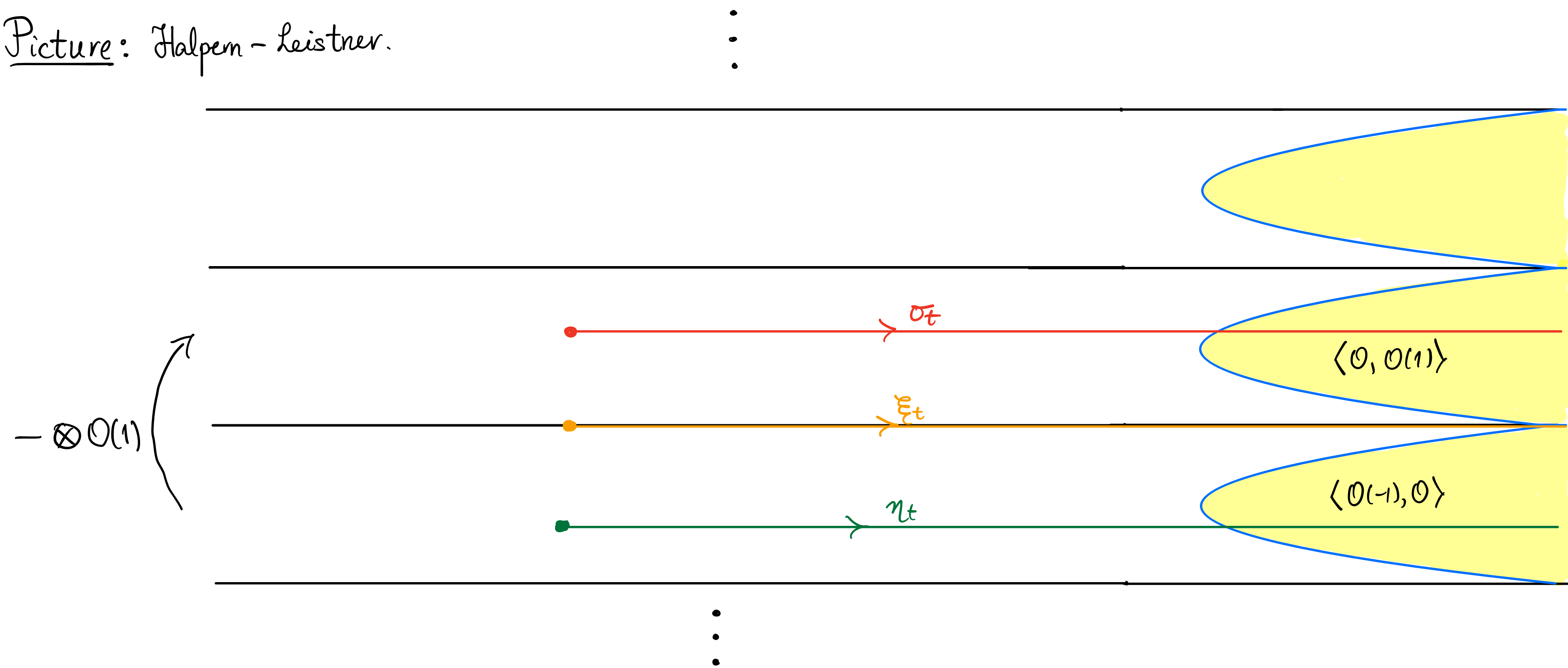
The case of $\mathbf{P}^1$ gives a good overview of general phenomena:

$\mathrm{Stab}(\mathbb{P}^1)/\mathbb{C} \cong \mathbb{C}$ (Okada)

Picture: Halpern-Leistner.



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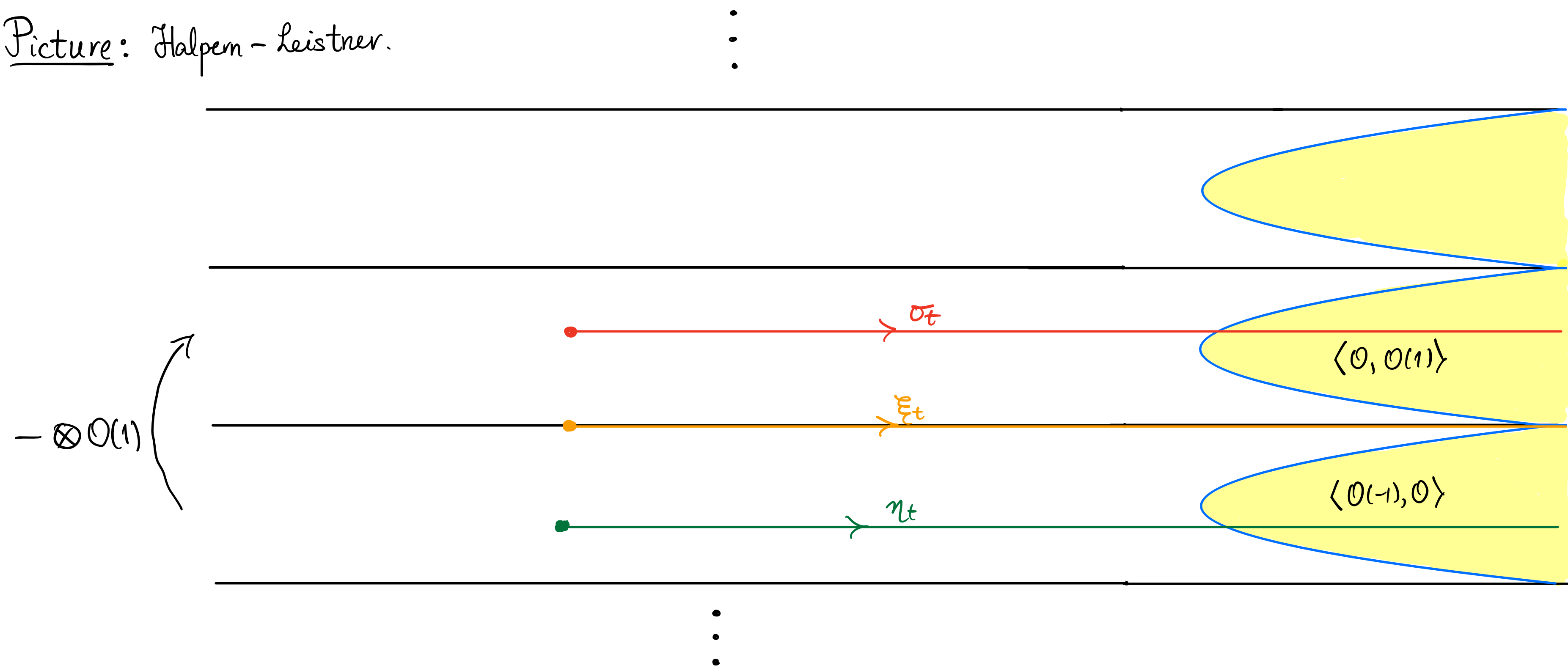

$\sigma_t \rightsquigarrow \langle \mathcal{O}, \mathcal{O}(1) \rangle$

$\eta_t \rightsquigarrow \langle \mathcal{O}(-1), \mathcal{O} \rangle$

$\xi_t \rightsquigarrow$??

Coordinates on the stability manifold

1. Given $\sigma \in \mathrm{Stab}(X)$, choose σ-stable $E_1, \ldots, E_n$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that $\{\mathrm{ch}(E_i)\}_{i=1}^n$ is basis of $H^*_{\mathrm{alg}}(X; \mathbf{C})$.
2. Bridgeland's Theorem $\Rightarrow \tau \mapsto (Z_\tau(E_1), \ldots, Z_\tau(E_n)) \in (\mathbf{C}^*)^n$ is a coordinate system around σ.
3. Put $\log Z_\tau(E_i) := \log|Z_\tau(E_i)| + i\pi\phi_\tau(E_i)$.

$$\tau \mapsto (\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \in \mathbf{C}^n$$

   logarithmic coordinates
4. $\forall\, w \in \mathbf{C}$, $\log Z_{w\cdot\tau}(E_i) = \log Z_\tau(E_i) + w$ so

$$(\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \mapsto (\log Z_\tau(E_1) + w, \ldots, \log Z_\tau(E_n) + w)$$

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Multiscale lines

$$\mathbf{C}^n/\mathbf{C} \iff \{(\mathbf{P}^1, \infty, dz, p_1, \ldots, p_n) \mid p_i \neq \infty \,\forall\, i\}/\cong$$

Proof:

- $\mu \in \mathrm{Aut}(\mathbf{P}^1)$: $\infty \mapsto \infty \Rightarrow \mu(z) = az + b$
- $\mu^*(dz) = dz \Rightarrow a = 1$

We compactify $\mathbf{C}^n/\mathbf{C}$ by introducing a "new" moduli space of marked genus 0 curves, called *multiscale lines* (inspired by Bainbridge - Chen - Gendron - Grushevsky - Möller).

Note: dz on $\mathbf{P}^1$ is characterised up to a scalar as a meromorphic differential with an order 2 pole at ∞; i.e, $dz \in \Gamma(\Omega_{\mathbf{P}^1}(2\infty)) = \mathbf{C}$.

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Definition (Part I)

An n-marked *multiscale line* is $(\Sigma, p_\infty, \preceq, \omega_\bullet, p_1, \ldots, p_n)$ where

1. Σ is a nodal genus 0 curve over $\mathbf{C}$
2. p_∞ is a special "top" point on the curve
3. $\preceq$ is a level structure
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+ conditions: e.g. $p_1, \ldots, p_n$ are all on components furthest from p_∞ and all such components are at the same "level" ...

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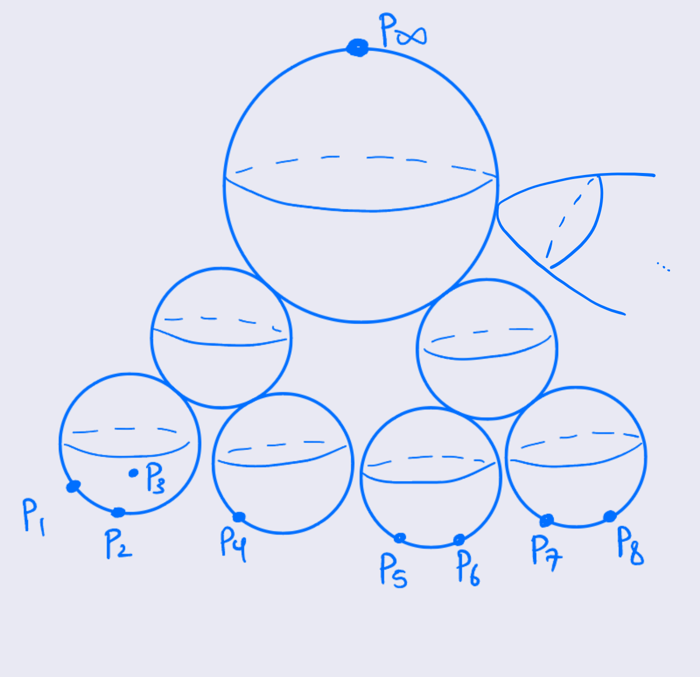

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Some Combinatorial types:

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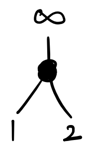

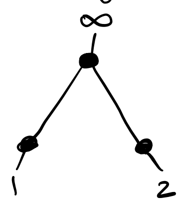

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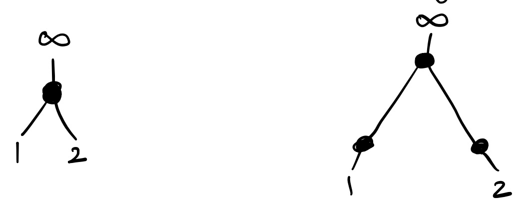

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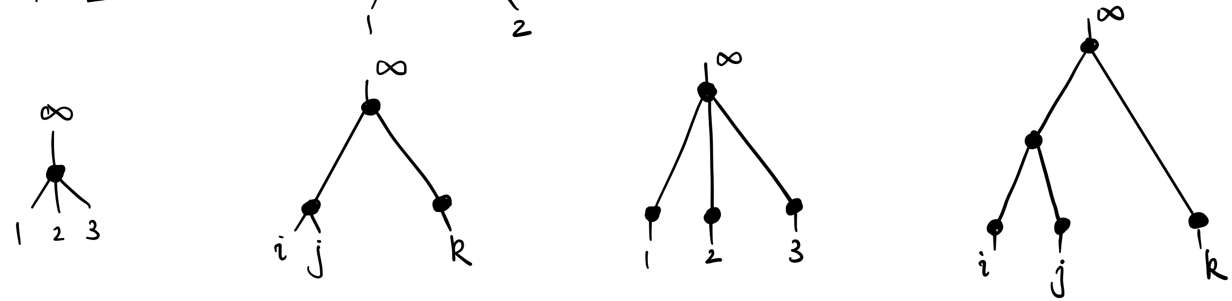

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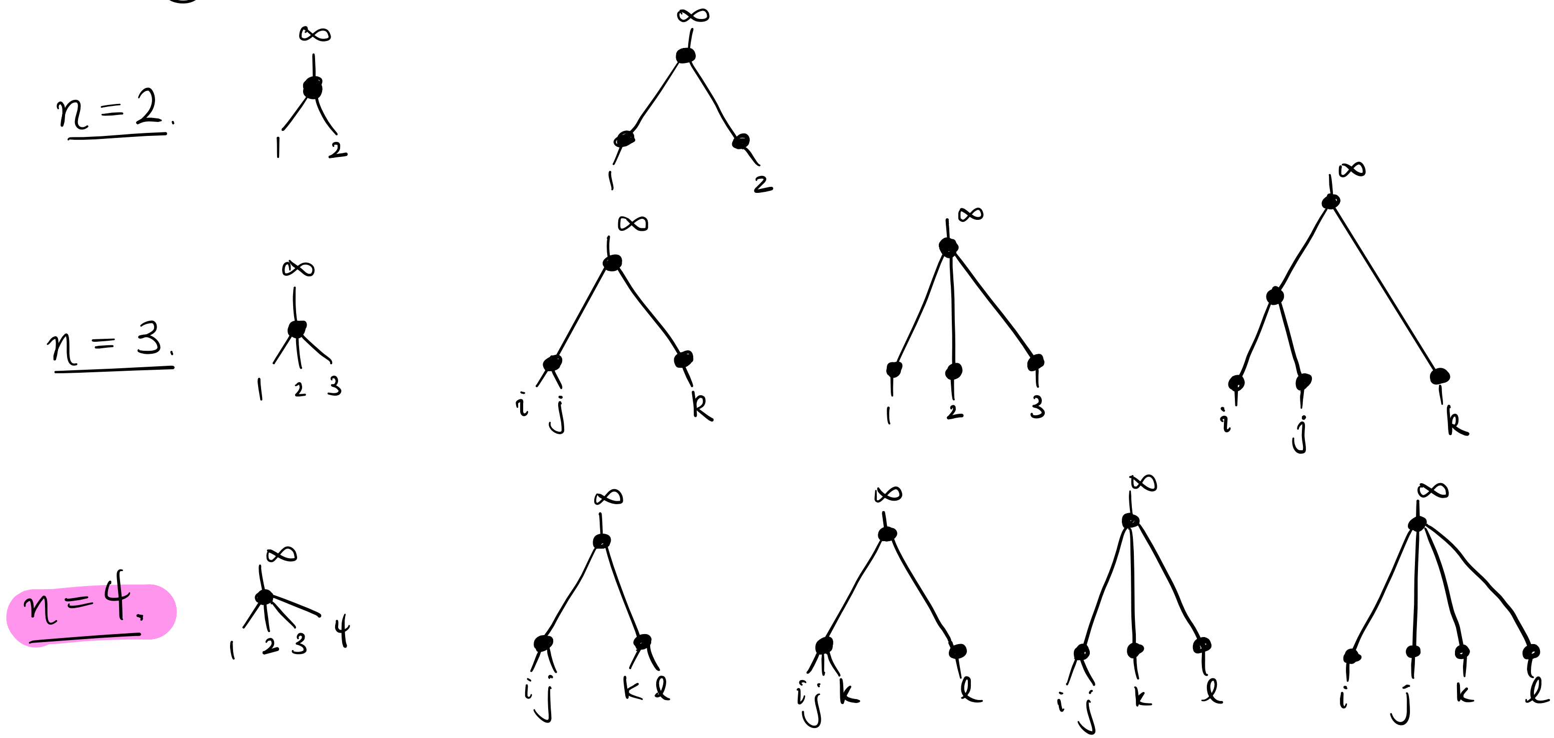

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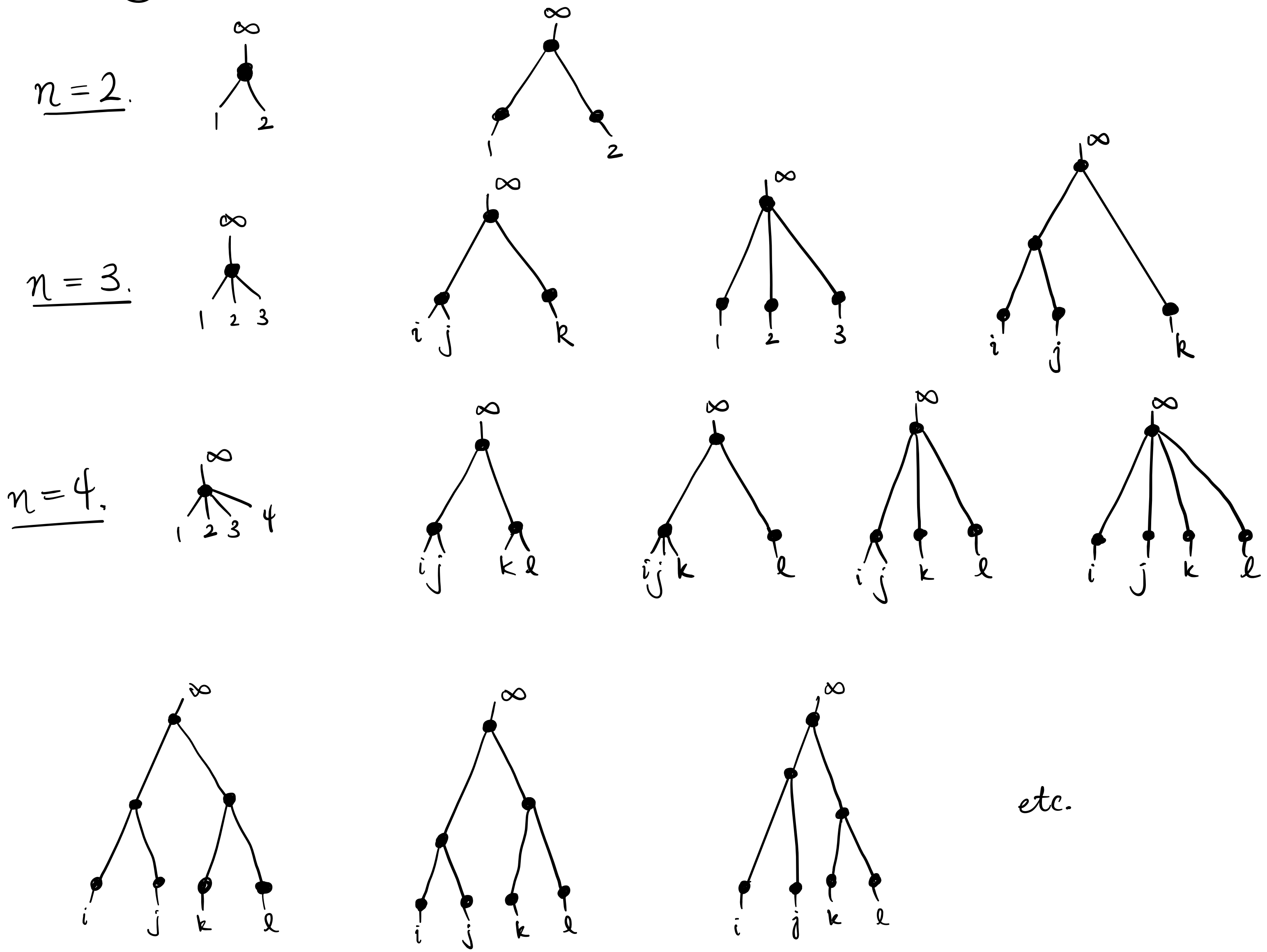

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... and $\omega_\bullet = \{\omega_v\}_{v \in V(\Sigma)} \rightsquigarrow$ meromorphic 1-forms, one per component Σ_v, with unique order 2 pole at the node connecting Σ_v to the root (resp. at p_∞ if $v = \text{root}$)

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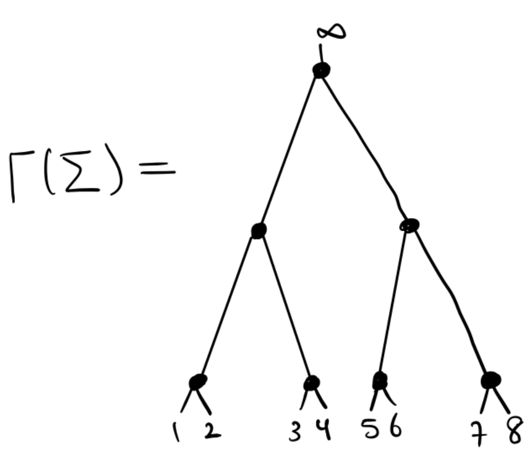

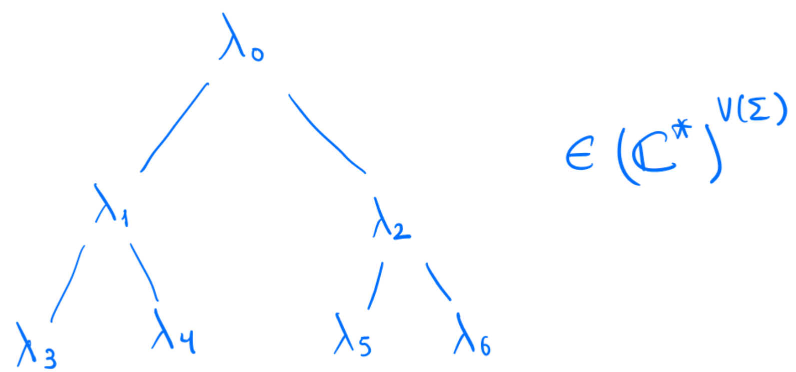

Equivalence

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A **C**-projective (resp. **R**-oriented) iso. of multiscale lines $f : \Sigma \to \Sigma'$ is an iso. of curves that preserves level structures and marked points s.t.

$$f^*(\omega'_v) = c_v \omega_v \quad \text{for } c_v \in \mathbf{C}^* \text{ (resp. } \mathbf{R}_{>0})$$

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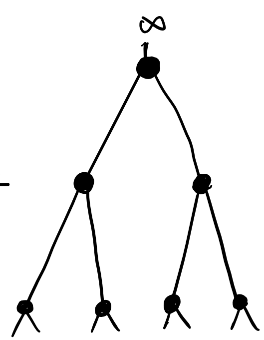
$\Gamma(\Sigma) =$ 			

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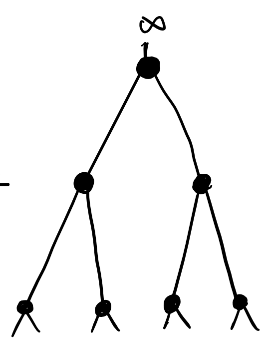
$\Gamma(\Sigma) =$ 	$\lambda_0 \in \mathbb{C}^*$ $(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$ $(\lambda_3, \lambda_4, \lambda_5, \lambda_6) \in (\mathbb{C}^*)^4$		

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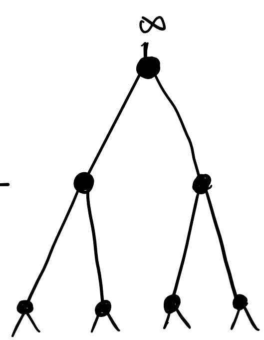
$\Gamma(\Sigma) =$ 	$(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$	$[\lambda_1 : \lambda_2] \in \mathbb{P}^1$	
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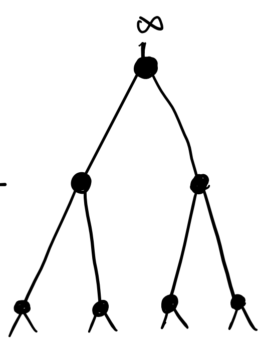
$\Gamma(\Sigma) =$ 	$\lambda_0 \in \mathbb{C}^*$	$[\lambda_0] \in \mathbb{P}^0$	$\lambda_0 \in \mathbb{C}^*/\mathbb{R}_{>0} = S^1$
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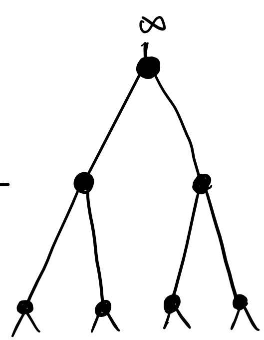
$\Gamma(\Sigma) =$ 	$(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$	$[\lambda_1 : \lambda_2] \in \mathbb{P}^1$	$(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2/\mathbb{R}_{>0}$
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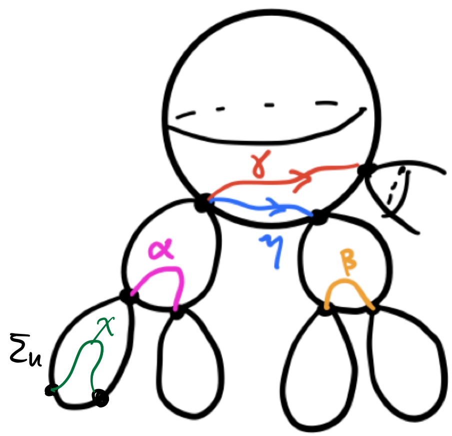
(All three notions agree in the irred. case.)

Examples



Up to $\mathbb{C}$ – projective isomorphism . . .

$$\int_{\chi} \omega_u , \quad \frac{\int_{\gamma} \omega_{root}}{\int_{\eta} \omega_{root}} , \quad \frac{\int_{\beta} \omega_w}{\int_{\alpha} \omega_v}$$

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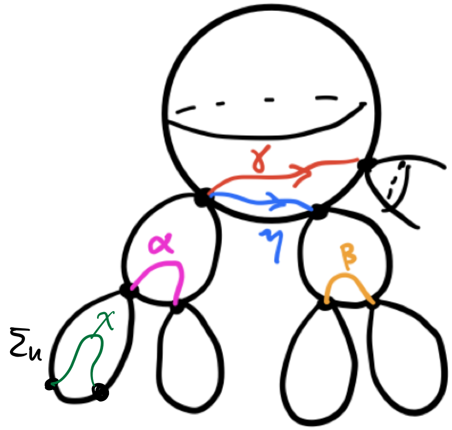

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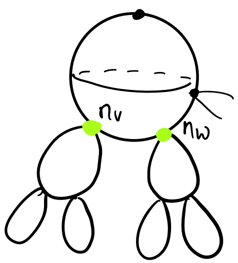
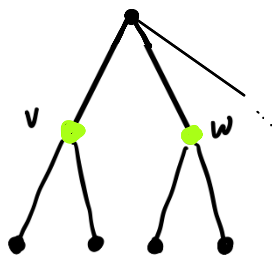
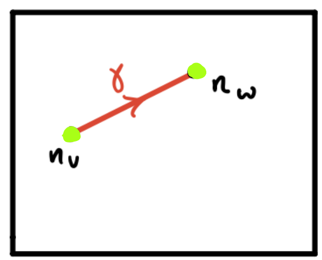
Up to $\mathbb{R}$-oriented isom., more functions are defined







$\Sigma_{root} \setminus \{p_\infty\}$

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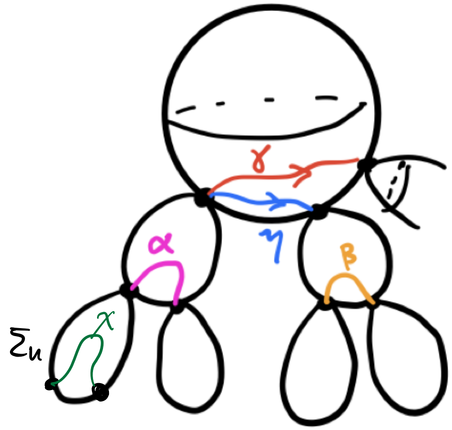
Examples



Up to ℂ-projective isomorphism...

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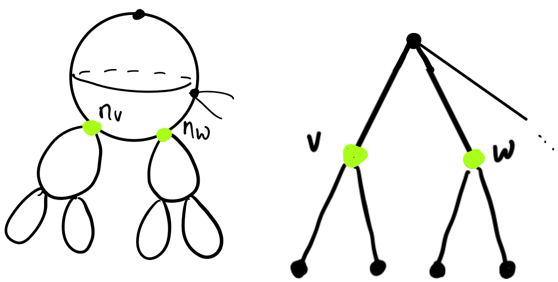

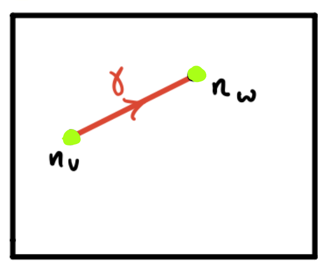

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Heuristic

A multiscale line Σ up to real-oriented isomorphism gives its dual tree the structure of a level graph with *angles* between edges

Moduli spaces

1. $\mathcal{A}_n := \{\mathbf{C} - \text{proj. iso. classes of } n\text{-marked multiscale lines}\}$
2. $\mathbf{C}^n/\mathbf{C} = \mathcal{A}_n^\circ \subset \mathcal{A}_n$ is the set of irreducible multiscale lines
3. Coordinates on $\mathcal{A}_n$ are constructed using the integral functions from the last slide.

Theorem (Halpern-Leistner, R.)

$\mathcal{A}_n$ is a compact complex algebraic manifold containing $\mathbf{C}^n/\mathbf{C}$ as an open dense subset. The boundary $D := \mathcal{A}_n \setminus \mathcal{A}_n^\circ$ is snc.

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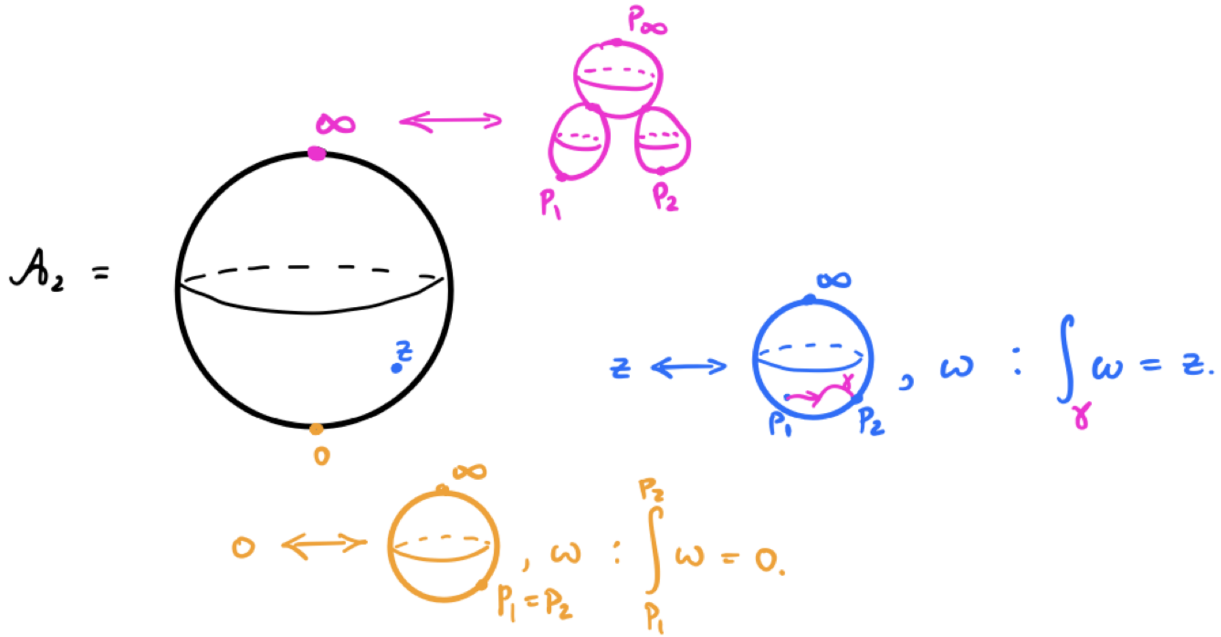

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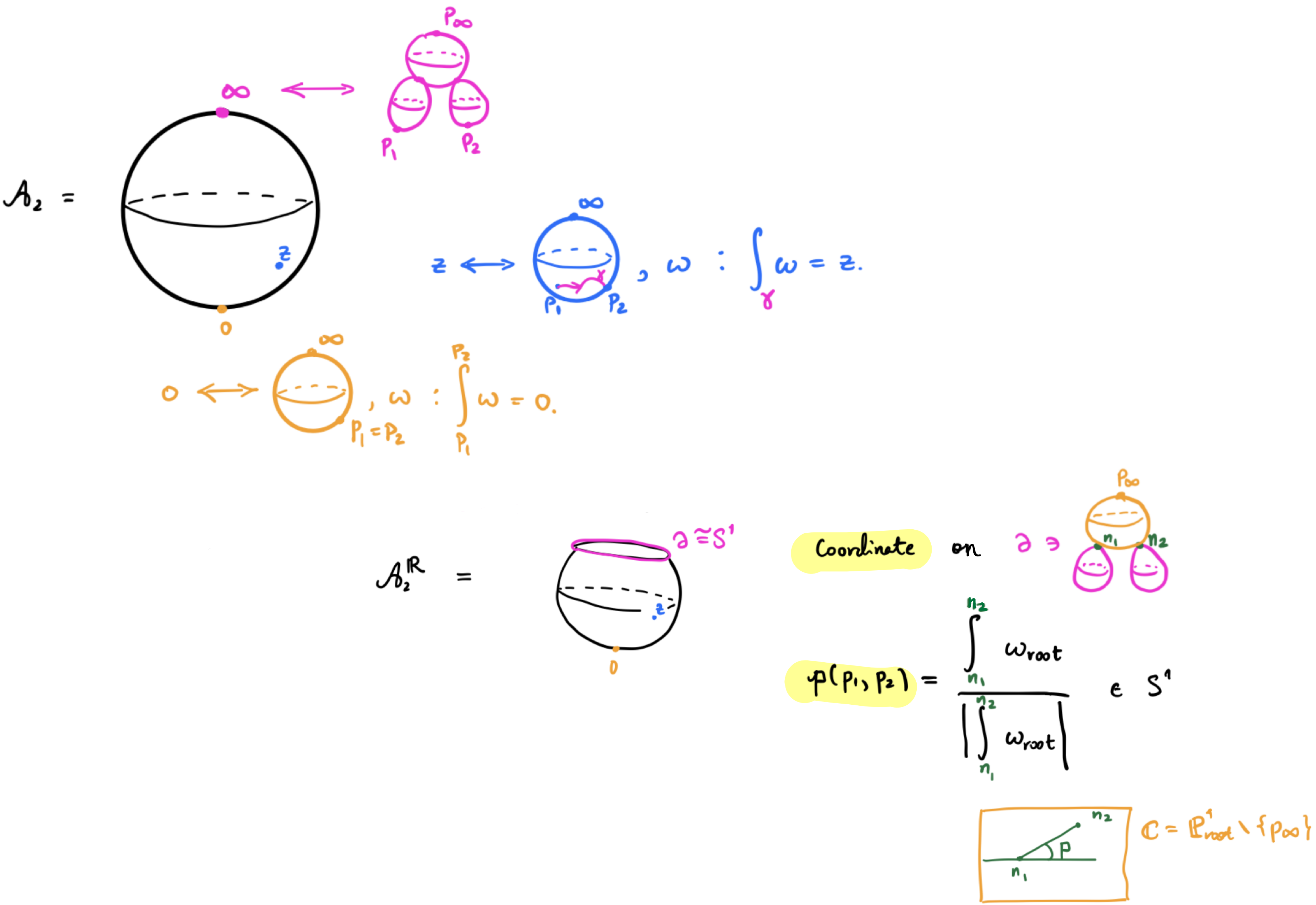

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A *multiscale decomposition* $\mathcal{D} = \langle \mathcal{D}_\bullet \rangle_\Sigma$ is:

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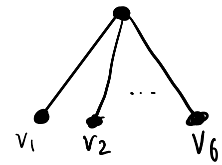
$\Gamma(\Sigma) =$ 

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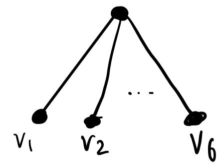
$\Gamma(\Sigma) =$ 

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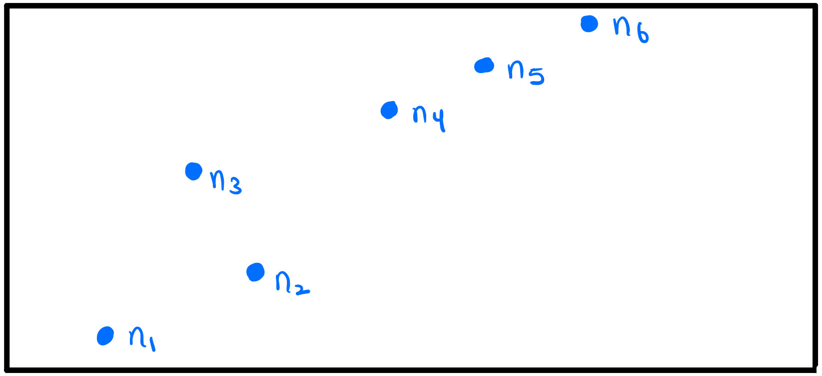

$\Im\mathfrak{p}(v_i, v_j) > 0 \Rightarrow \mathrm{Hom}(\mathcal{D}_{\leq v_j}, \mathcal{D}_{\leq v_i}) = 0$; get sod $\mathcal{D} = \langle \mathcal{D}_{\leq v_1}, \ldots, \mathcal{D}_{\leq v_6} \rangle$.

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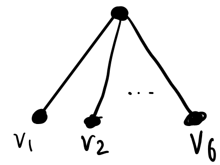

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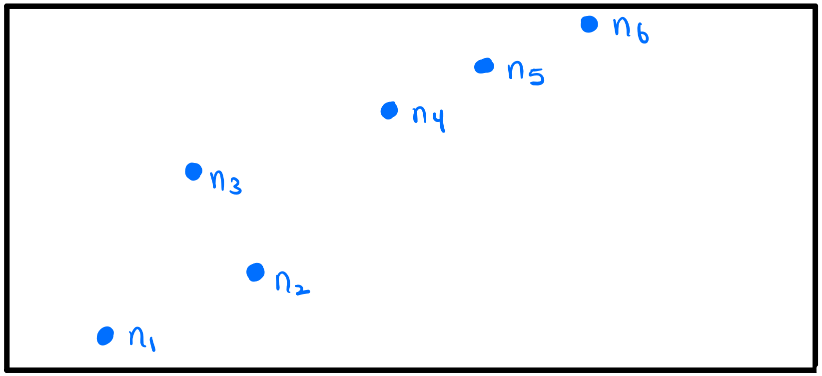

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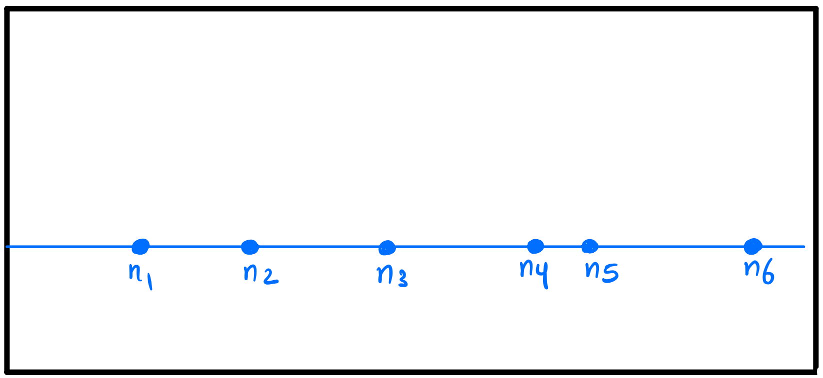

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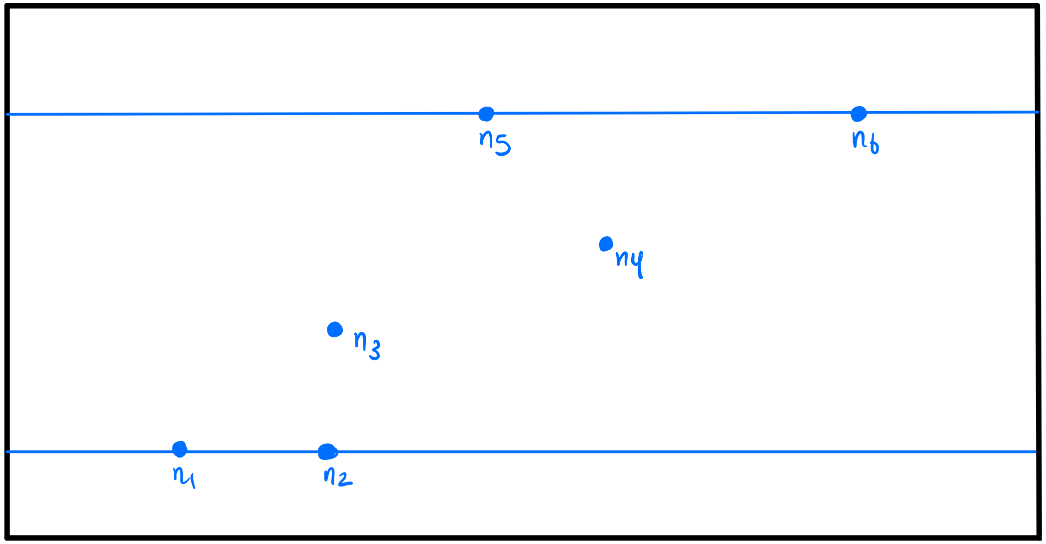

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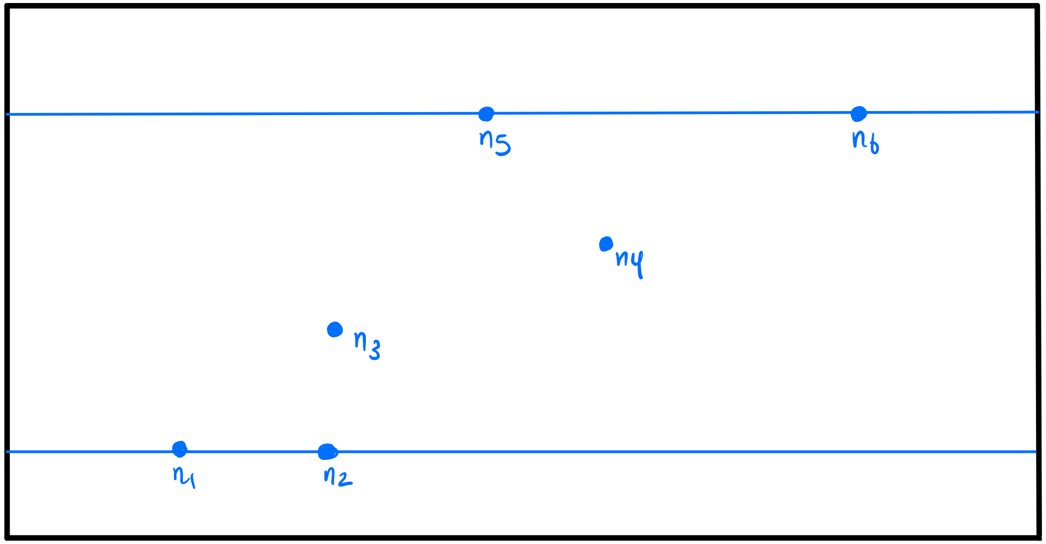

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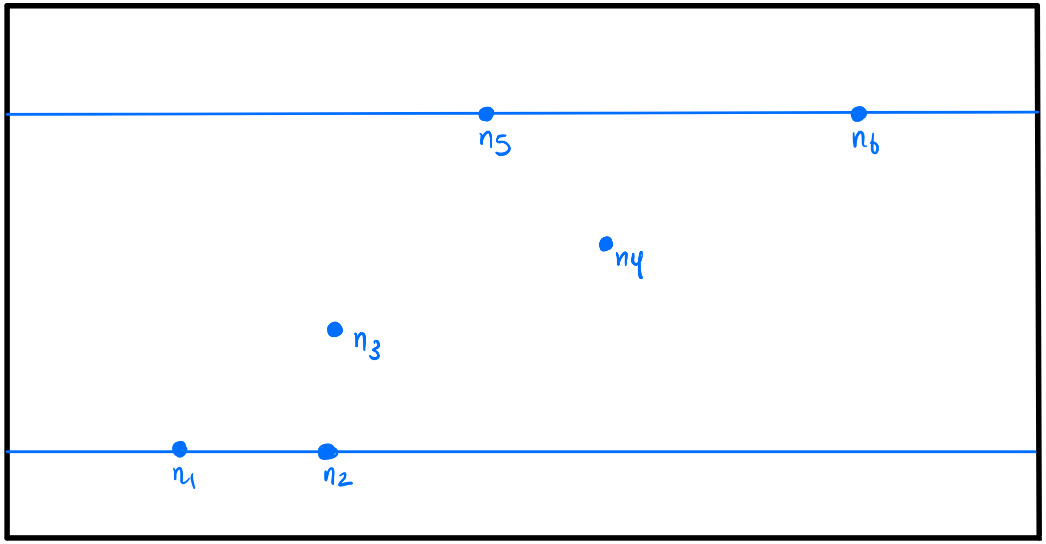

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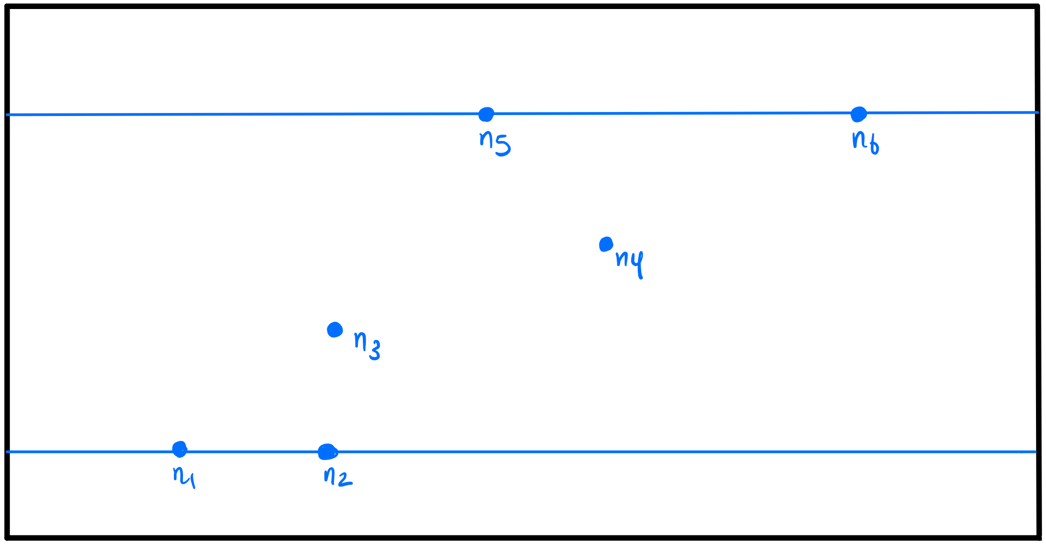

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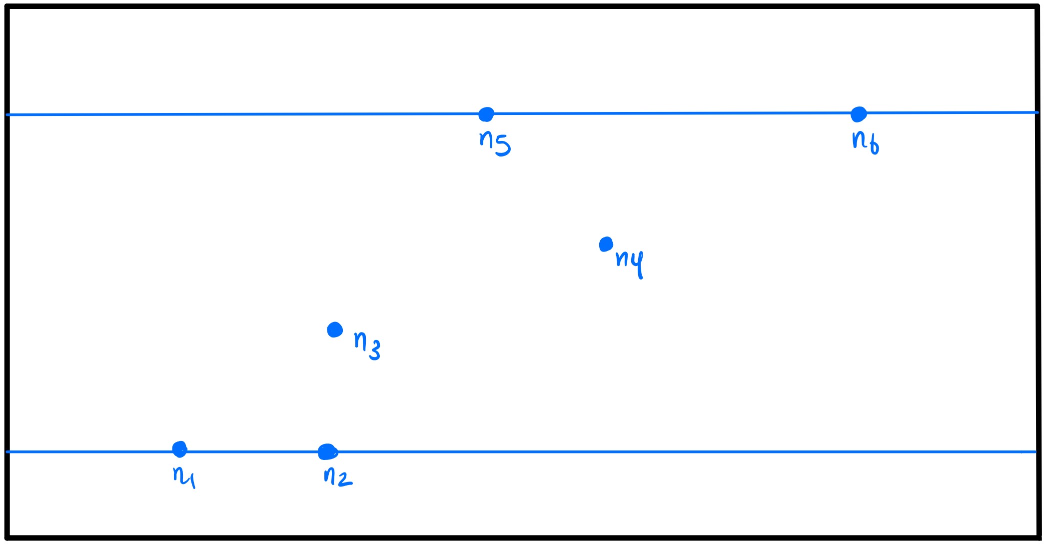

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Augmented stability conditions

Definition (Augmented stability condition)

An *augmented stability condition* $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ is a multiscale decomposition $\mathcal{D} = \langle \mathcal{D}_\bullet \rangle_\Sigma$ such that $\mathrm{gr}_v(\mathcal{D}_\bullet)$ is equipped with $\sigma_v \in \mathrm{Stab}(\mathrm{gr}_v(\mathcal{D}_\bullet))/\mathbf{C}$ for each $v \in V(\Sigma)_{\mathrm{bot}}$.

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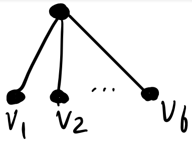

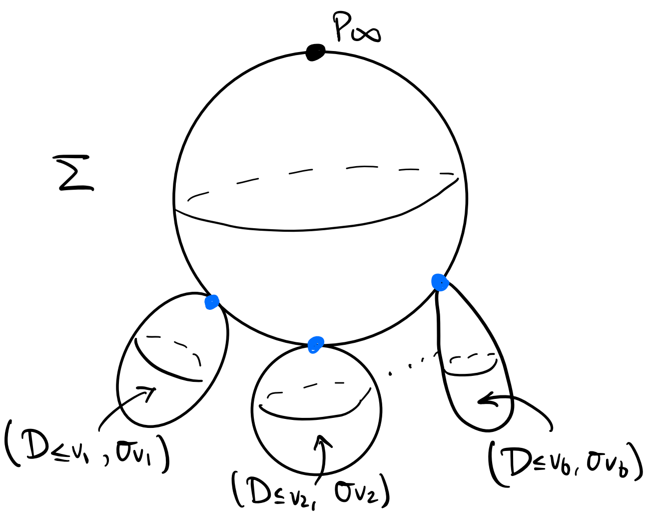

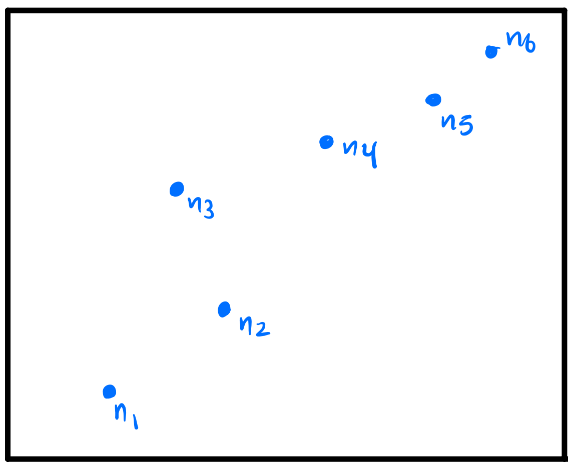

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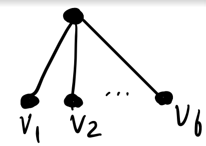

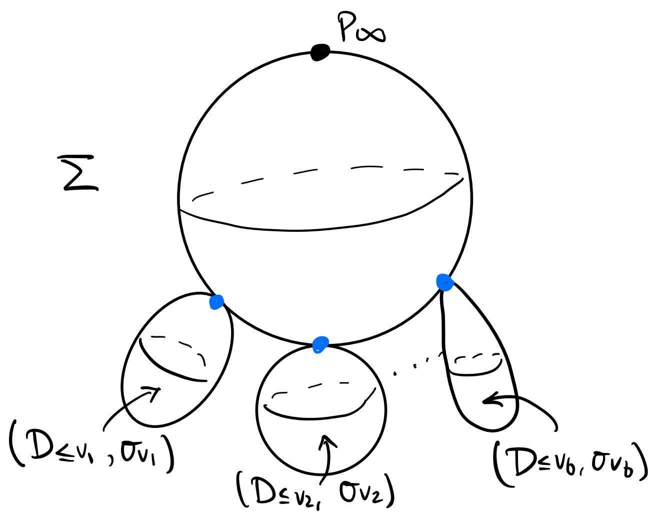

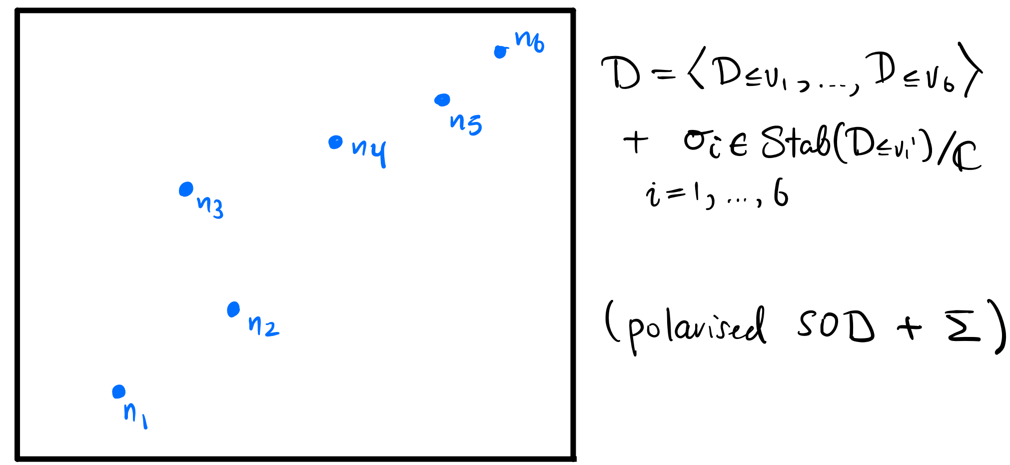

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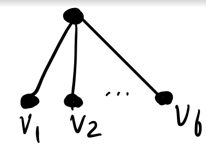

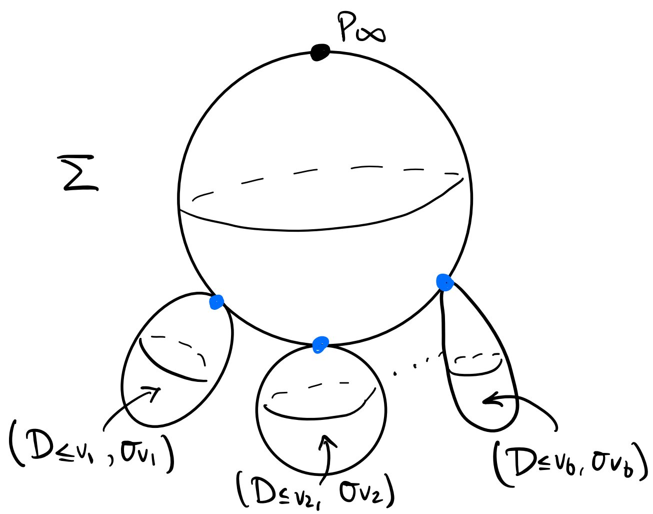

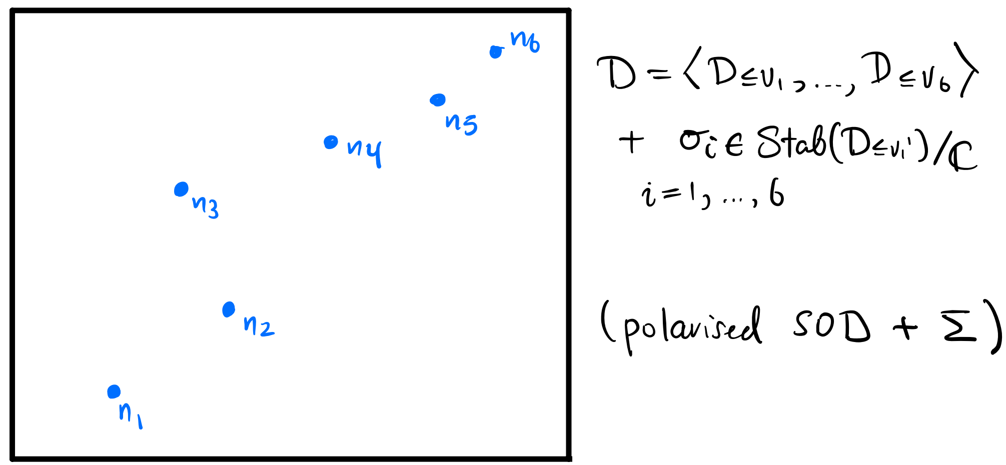

$\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ is identified with the set of points in $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$ of the form

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Main Theorem

$\mathcal{A}\,\mathrm{Stab}(\mathcal{D}) :=$ set of augmented stability conditions.

Theorem (Halpern-Leistner, R.)

There is a Hausdorff topology on $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$ such that

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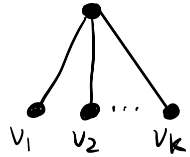
1. Take $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ with Σ generic. For simplicity, let $\Gamma(\Sigma) =$ 

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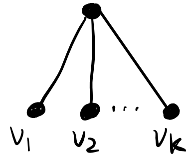
1. Take $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ with Σ generic. For simplicity, let $\Gamma(\Sigma) =$ 

2. $\mathcal{D} = \langle \mathcal{D}_{\leq v_1}, \ldots, \mathcal{D}_{\leq v_k} \rangle \rightsquigarrow H^*(\mathcal{D}) = \bigoplus_{i=1}^{k} H^*(\mathcal{D}_{\leq v_i})$.
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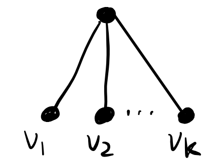
1. Take $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ with Σ generic. For simplicity, let $\Gamma(\Sigma) =$ 

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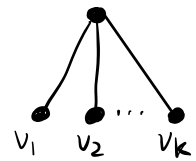
1. Take $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ with Σ generic. For simplicity, let $\Gamma(\Sigma) =$ 

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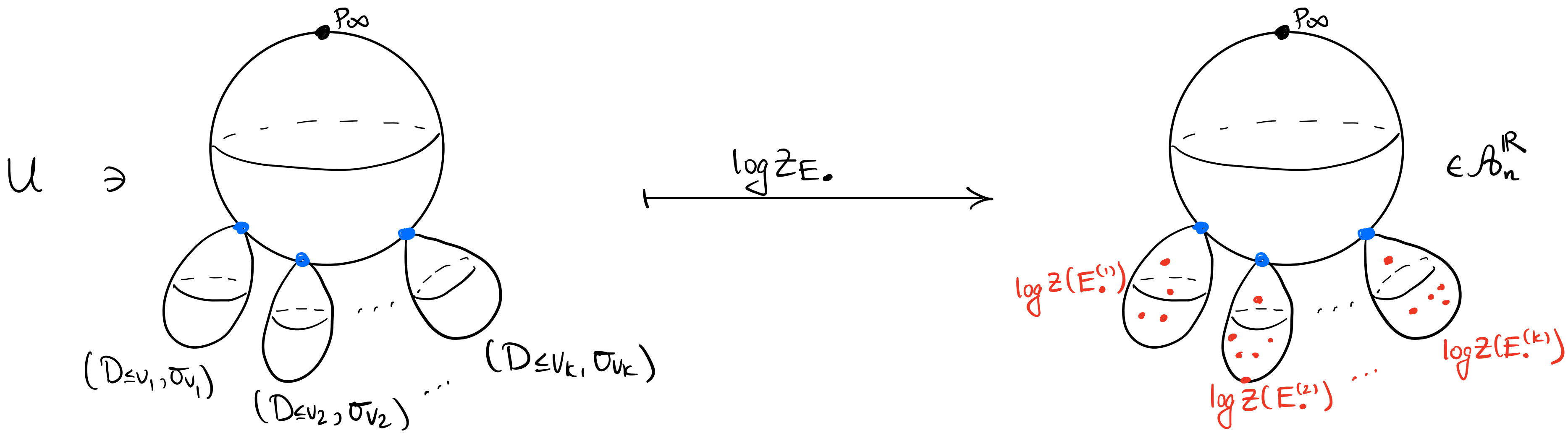
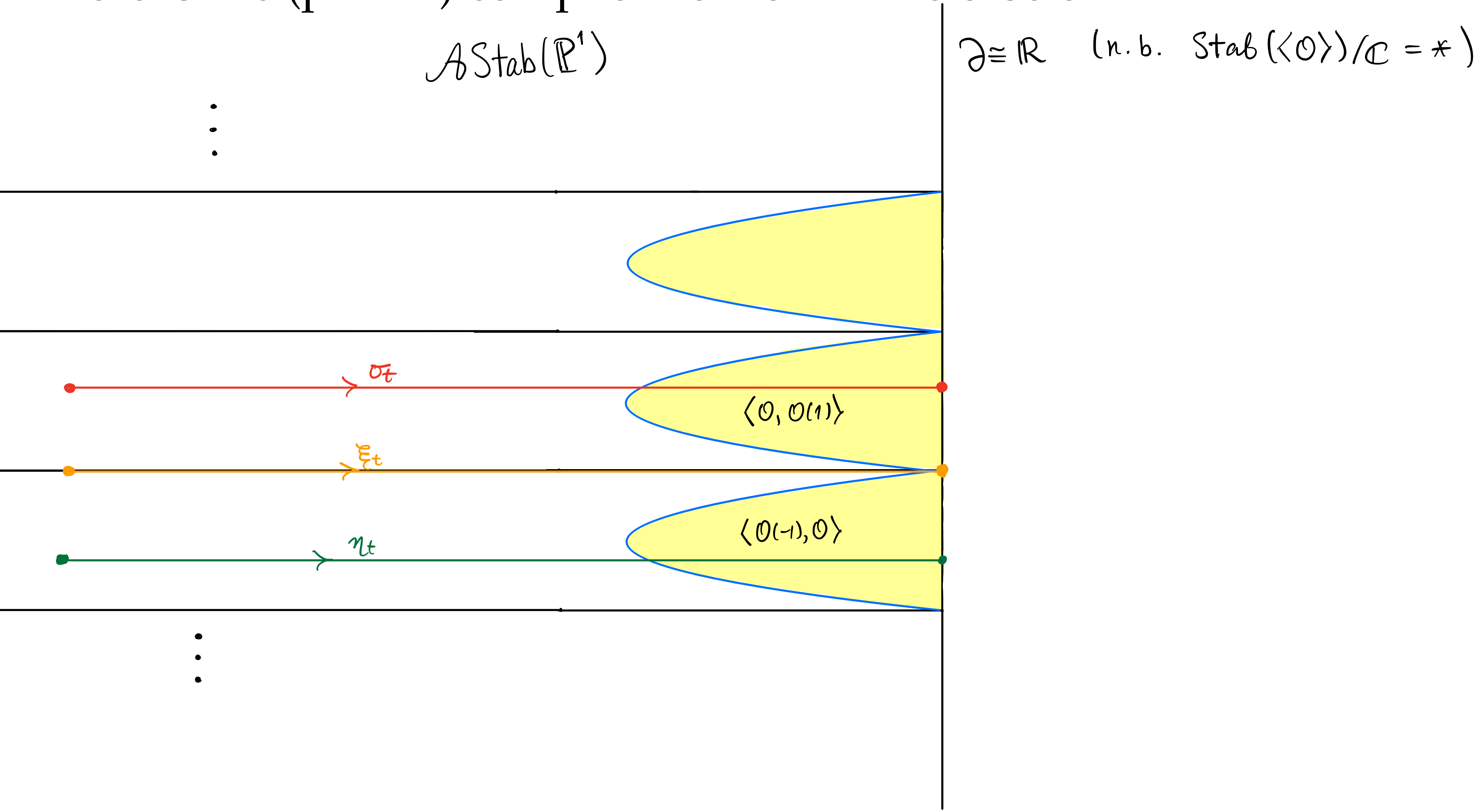
4. $\exists\, U \ni \sigma$ on which $E_\bullet^{(i)}$ are all stable and the map $\log Z_{E_\bullet} : U \to \mathcal{A}_n^{\mathbf{R}}$ is a local homeo.



$\mathbf{P}^1$ revisited

Here is the (partial) compactification in the case of $\mathbf{P}^1$.



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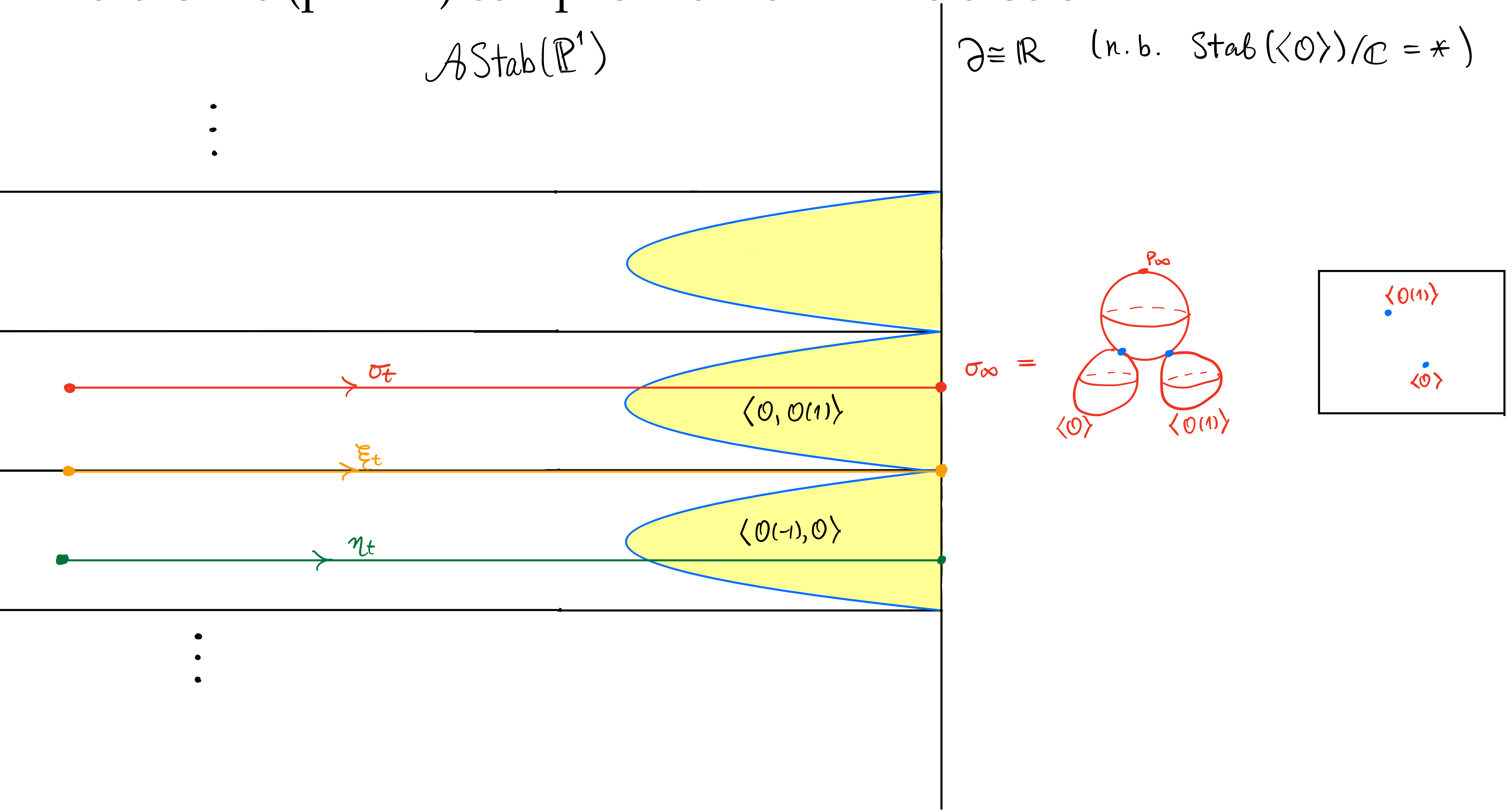

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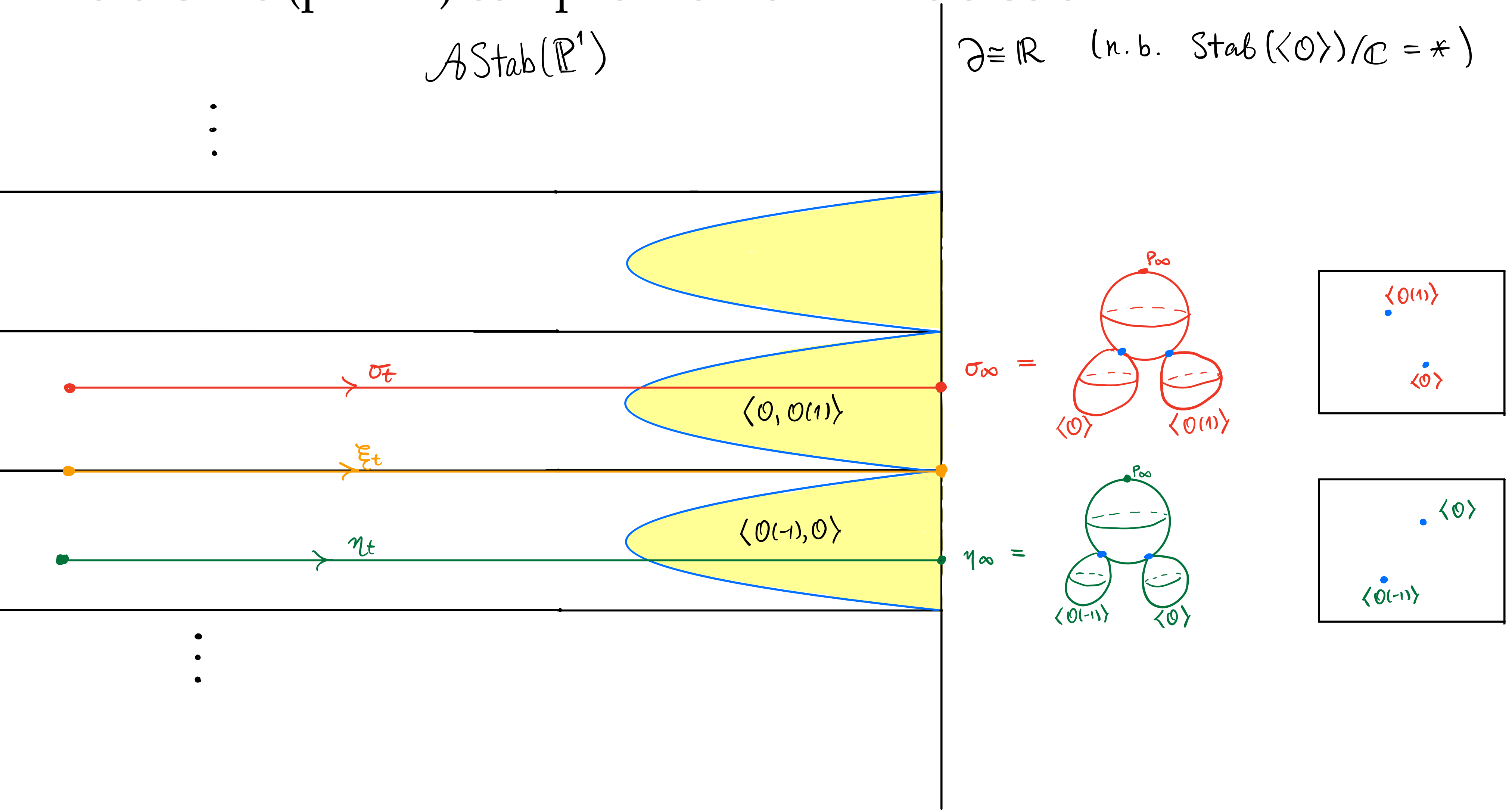

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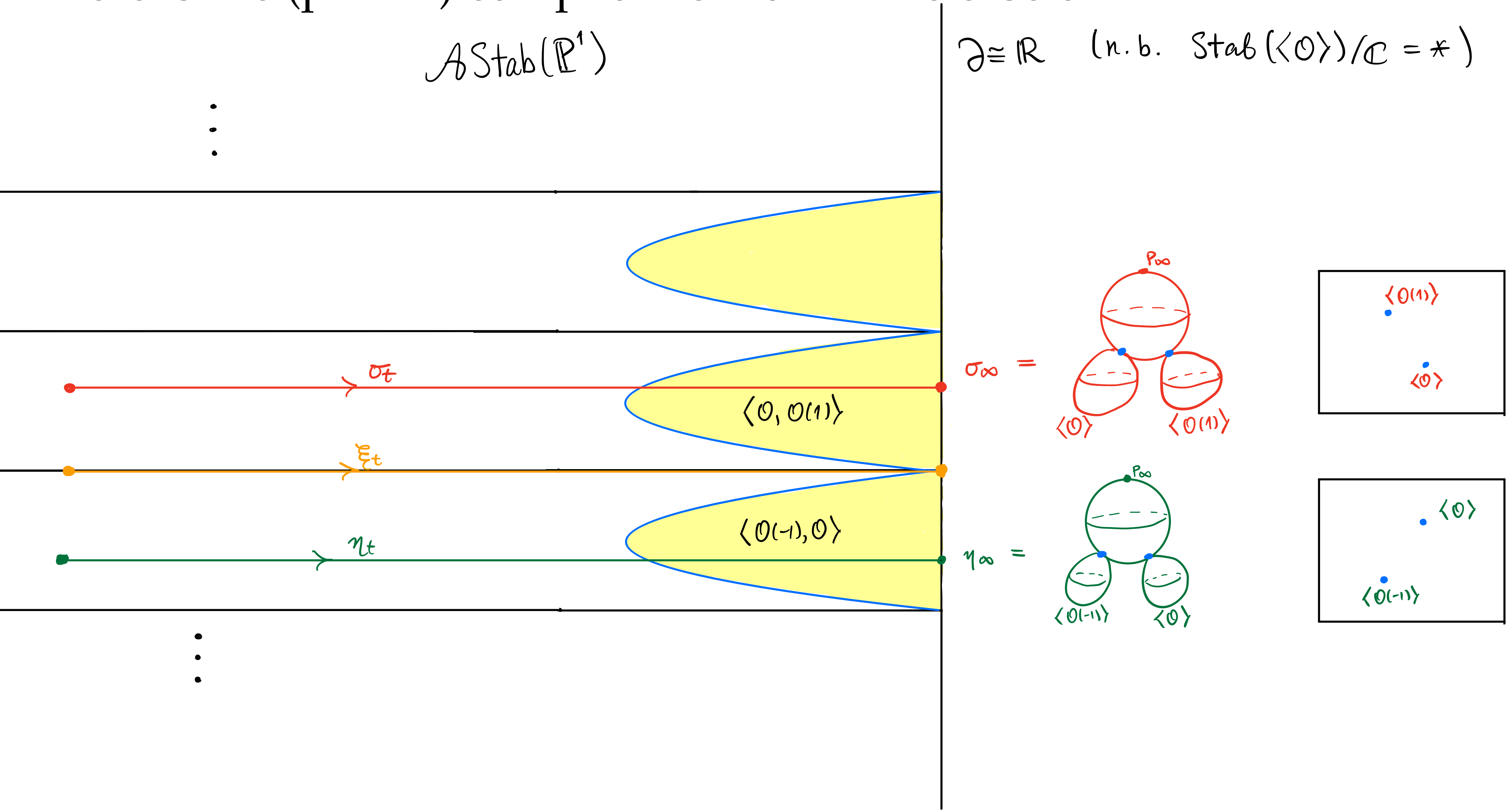
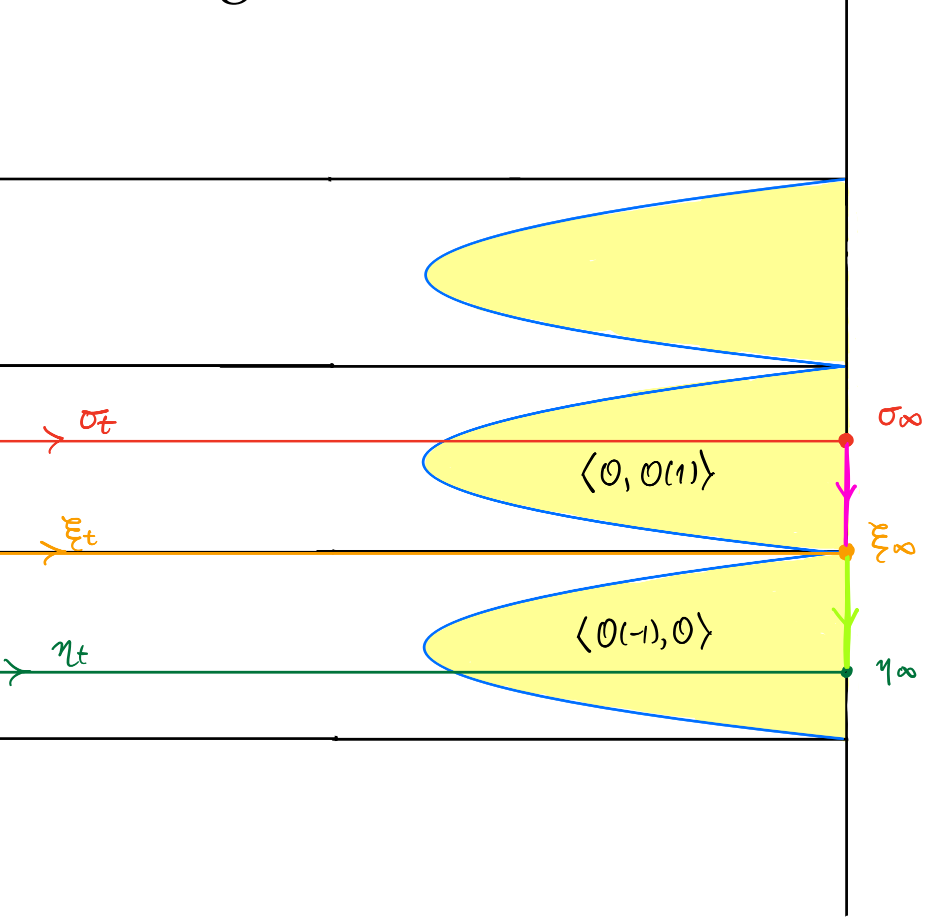
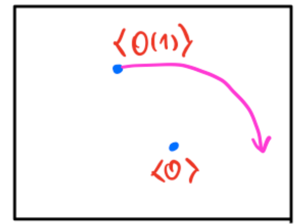


Question: what is the limiting point of ξ_t as $t \to \infty$?

Path along the boundary

We consider a path along the boundary from σ_∞ to η_∞, which passes through ξ_∞.





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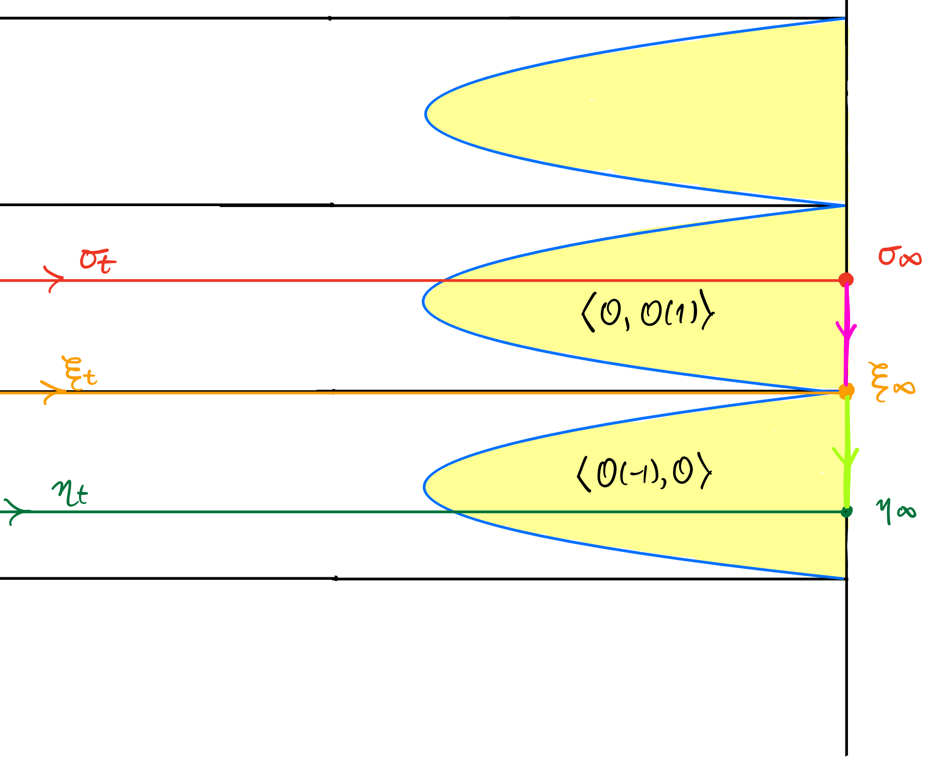

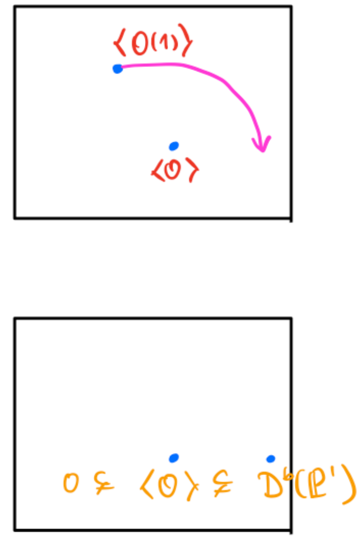

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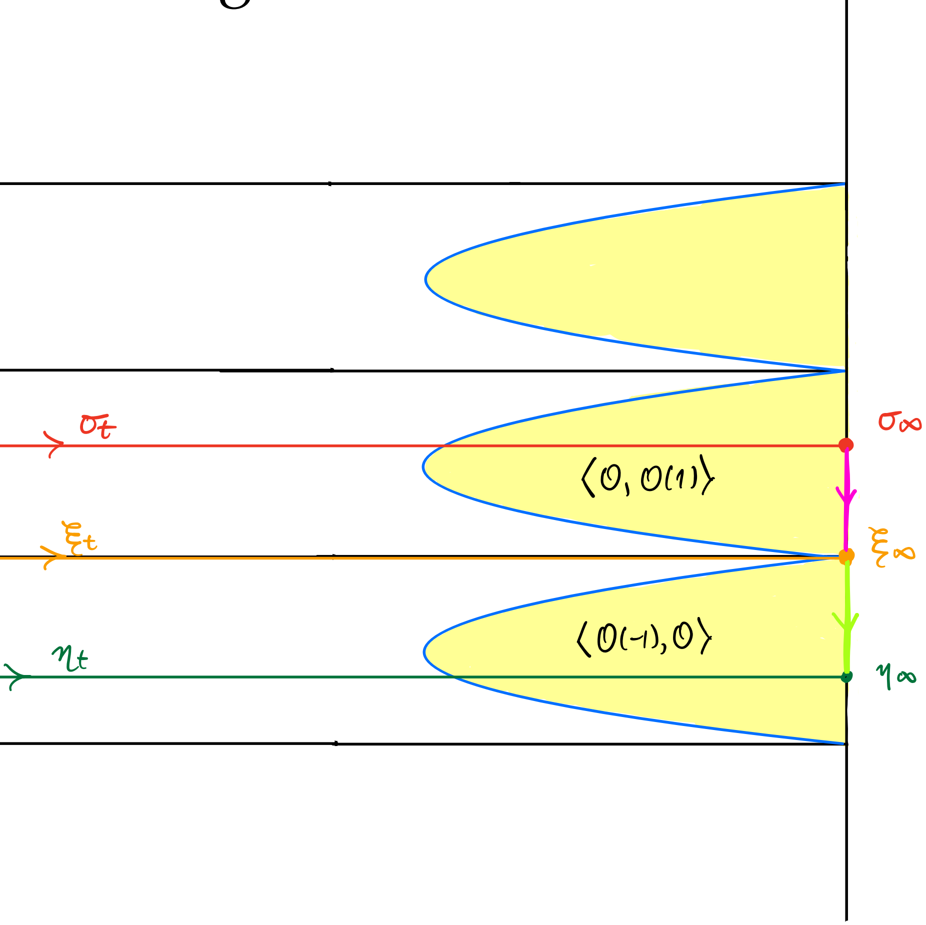

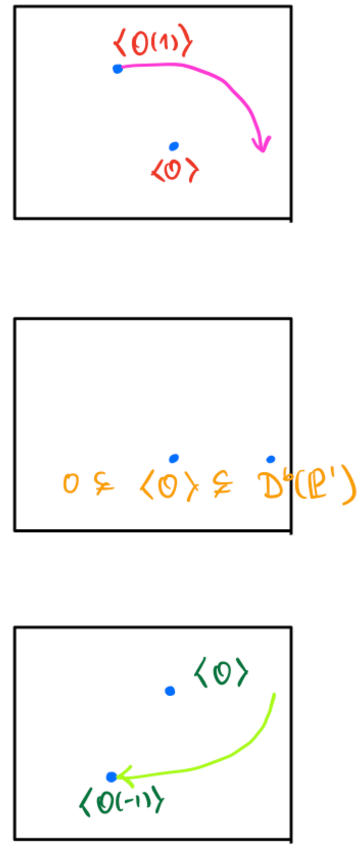

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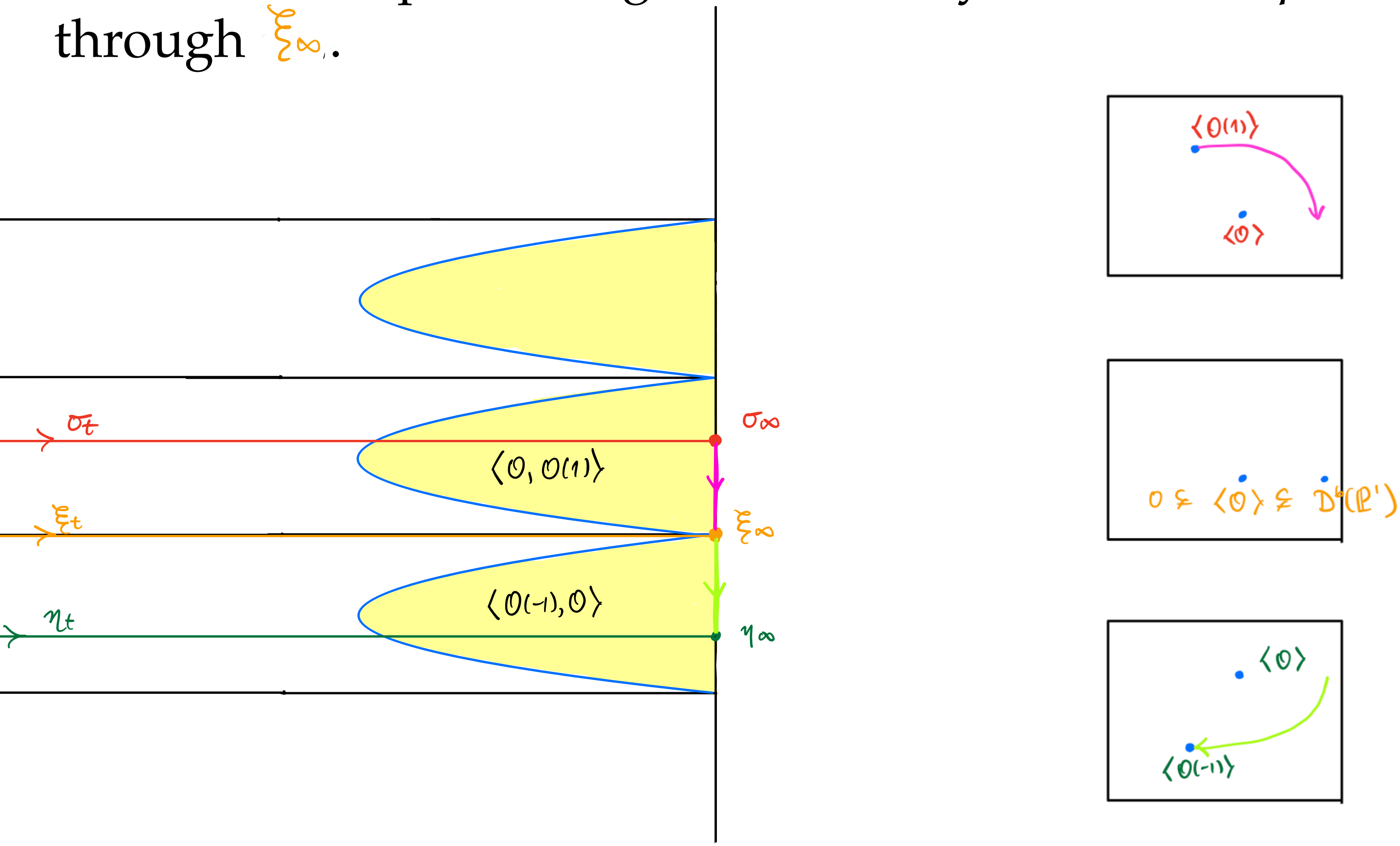

The boundary point $\lim_{t\to\infty} \xi_t$ is a *degenerate* semiorthogonal decomposition, i.e. an admissible *filtration* $0 \subsetneq \langle \mathcal{O} \rangle \subsetneq \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbf{P}^1)$.

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2.
3.
4.
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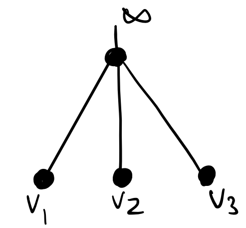

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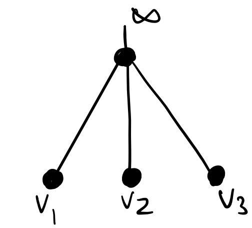

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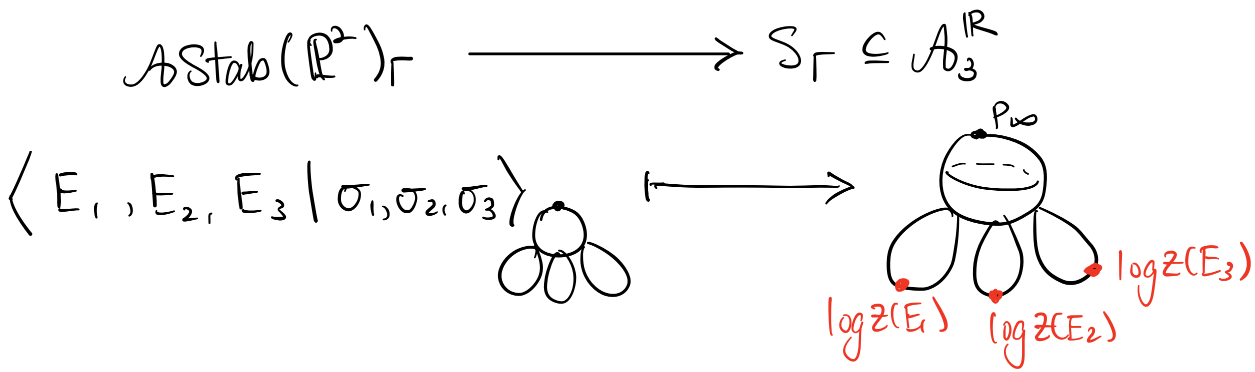

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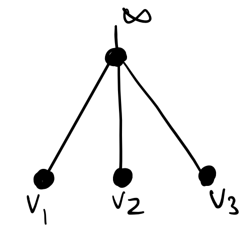

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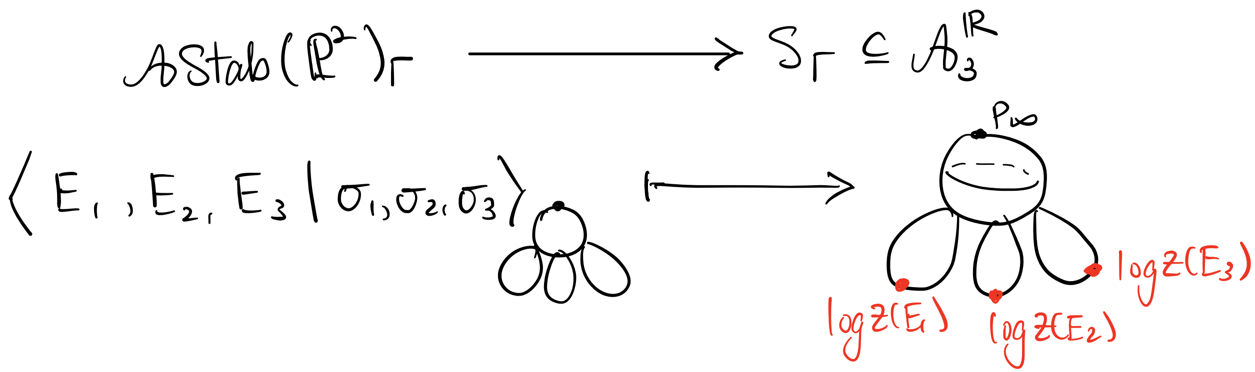

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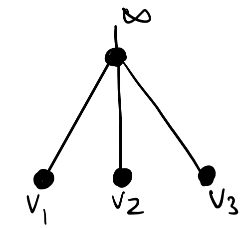

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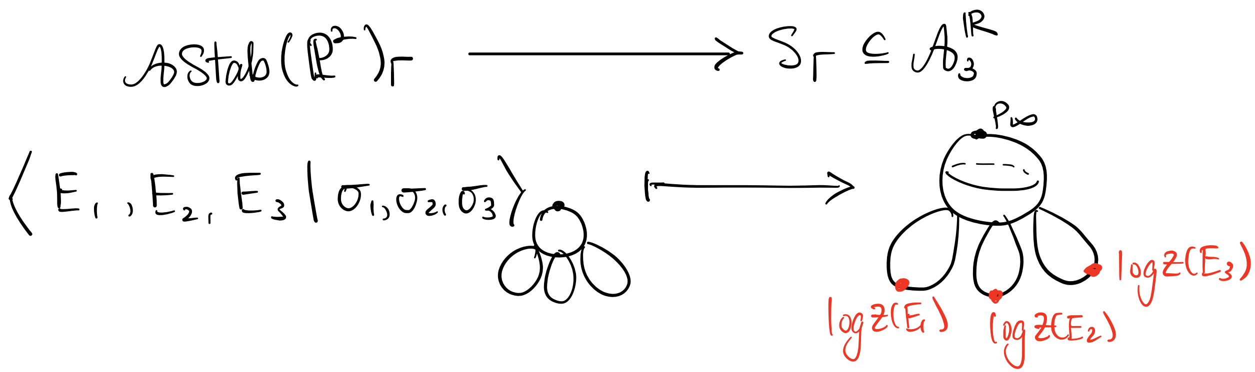

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$$\gamma \cdot \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle_\Sigma = \langle E_1, E_2, E_3 \rangle_\Sigma$$

   where E_1, E_2, E_3 is obtained by mutation along b.

Perspective

Proposition (Informal)

Connected components of strata in $\partial\mathcal{A}\,\mathsf{Stab}$ correspond to equivalence classes of SODs up to mutation.

This gives us a revised:

Heuristic

Given $\sigma_0, \tau_0 \in \mathrm{Stab}(X)/\mathbf{C}$ and corresponding paths σ_t and τ_t, one hopes σ_t and τ_t converge to points in the same connected component of $\partial\mathcal{A}\,\mathrm{Stab}(X)$, giving a canonical mutation class of SOD for $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

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Thank you!

Thank you for listening!